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ON THE TORSION RESULTING FROM FLEXURE IN PRISMS WITH CROSS-SECTIONS OF UNI-AXIAL SYMMETRY ONLY

BY

ANDREW W. YOUNG, M.A., ETHEL M. ELDERTON
AND KARL PEARSON, F.R.S.

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NOTE. This paper was originally written at my suggestion by Mr A. W. Young. A slip in the analysis (involving the recalculation of the whole of the arithmetical work) was pointed out by Mr Arthur Berry, who examined the paper in proof. I am deeply grateful to him for the discovery. Owing to Mr Young's call to military service he has been unable to reconstruct the paper. Accordingly for the paper in its present form Miss Elderton and I are responsible, not only for the calculations, but for the much modified and considerably augmented analysis. K. P.

The Torsion resulting from Flexure in Prisms with Cross-sections of Uni-axial Symmetry.

Introduction

The problem of the flexure of a beam under the action of a single load stands today in much the same state as Saint-Venant left it half a century ago, nearly all of the more recent work having been devoted to the problem of flexure with *continuous* systems of loading. There has not even been any notable additions to the forms of the cross-section of the beam discussed by Saint-Venant and the rectangular and elliptical cross-sections may be taken as the only cross-sections of simple mathematical form for which full solutions exist. It appears, for instance, that, although the general solution of the problem on Saint-Venant's assumptions includes the case of a beam having a uni-symmetric cross-section, no such case has been discussed analytically, far less numerically. Yet the case of general asymmetry in the cross-section possesses peculiar interest and has recently become of real importance in connection with the blades of aeroplane propellers. With the high speed of rotation to which the aeroplane propeller attains there is a strong flexing force in the air resistance and, as we shall see, the flexure of a beam of asymmetrical cross-section produces a torsion of a by no means negligible amount. To the designer of the aeroplane propeller this is of the utmost importance for even a slight change in the 'angle of attack' of a propeller-blade might seriously alter the efficiency and speed of the machine. It may well be that more potent causes exist for the torsion of a propeller-blade subjected to a continuous system of pressure and not merely to a single end-load, but to gain some knowledge of the nature of the torsion produced by flexure in the simpler case of the present paper must be regarded as the first step towards the investigation of the more difficult problem. There is the further limitation to the generality of the discussion of the following pages that we shall treat only of cross-sections which have an axis of symmetry perpendicular to the plane of flexure. The problem of complete asymmetry still awaits investigation.

The torsion spoken of above which arises in the flexure of a beam of asymmetrical cross-section can in general terms be seen to arise from two contributory causes. Firstly there is the distortion of the cross-sections of the beam which Saint-Venant showed to be a necessary accompaniment of flexure. With an asymmetrical cross-section the distortion, it is clear, will be likewise asymmetrical.

If then we fix the end-section in almost* any way which will allow the distortion of the theory to take place, this distortion and particularly that part of it which accompanies the curvature, called by Thomson and Tait *anticlastic*, will produce a rotation of the beam as a whole, together with what we may call an 'impure' torsion. The latter we shall term the *torsion due to anticlastic curvature.* It may be noted in passing that even in the case of a beam of symmetrical cross-section, there will arise a rotation of this nature if the end fixing be asymmetrical in character.

This torting effect of the distortion of the sections is however only one portion of the total torsion; with it must be combined the effect due to the distribution of shears† in every cross-section of the beam—a distribution which is necessarily asymmetrical if the cross-section be asymmetrical. It is the main object of this paper to investigate this *torsion due to shear* with more particular reference to the case where the cross-section is a circular sector, complete or curtate. The solution of the corresponding torsion problem exists for this cross-section, and with numerical reductions such as we should expect in work done by Saint-Venant‡. Arithmetical work is even more necessary in the flexure problem and no small amount of care has been devoted in this paper to make clear the analysis by numerical calculation, whenever that was possible. In addition, more perhaps with a view to strengthening the authors' faith in the theoretical results than to proving them to the complete satisfaction of others, a series of rough experimental tests were carried out with various beams of circular sector cross-section. In the course of the discussion notes will be found of these observations. They are given for what they are worth and make no pretence to be the results of accurate laboratory experiments. More painstaking and less hurried practical investigation must be postponed to times of greater leisure.

§ 1. *Description of the contents of this Paper.*

Saint-Venant in his original Memoir§ of 1856 made use of the semi-inverse or 'mixed' method of treatment and assumed both that the stresses perpendicular to the longitudinal axis of a flexed beam across planes parallel to that axis were everywhere zero‖ and that the extension at any point was proportional to the distance of that

* An unimportant exceptional case will be pointed out later on, § 9.

† We shall use 'shear' in the sense of a stress in which it was originally introduced into elastical nomenclature by George Stephenson, the corresponding strain being called a 'slide' by Pearson. Cf. Todhunter and Pearson's *History*, Vol. I, p. 882.

‡ *Comptes Rendus*, T. LXXXVII, 1878, pp. 849—54 and 893—9. The paper is described in Todhunter and Pearson's *History*, Vol. II, Part I, p. 191 *et seq.* Compare also Greenhill, *Mess. of Maths.*, Vol. 10 (1880), p. 83.

§ *Mémoire sur la flexion des prismes*, *Journal de Liouville*, Tome 1[er] (2[e] série), pp. 89—189.

‖ Or, in other words, that there was *no mutual stress between the longitudinal 'fibres' of the beam perpendicular to those fibres.* This is the phrase used by most writers, Saint-Venant among them, but the suggestion of fibrous material which it conveys makes it unsuitable for use without qualification although it brings the physical meaning of the hypothesis well before the imagination.

point from the neutral plane. Clebsch* showed that the second assumption flowed from the first, but, as his equations are all for the isotropic case, we have briefly in §§ 2—4 given the required equations generalised for the aeolotropic case which we treat later on in the paper.

In the second part of the paper (§§ 5—8) the general equations—isotropy being for the present assumed—are transformed for use with cylindrical coordinates and the complete analytical solution given for the case of a uniform beam with cross-section bounded by two concentric arcs and two radii, the plane of flexure being taken at right angles to the radial plane of symmetry of the beam. This plane of symmetry is therefore the 'neutral' plane as regards extension.

Then follows (§§ 9—14) an arithmetical reduction of the analysis, especially in connection with the torsion due to shearing stress and its dependence on the angle and on the curtateness of the sector. The results, both of theory and of experiment, lead to investigation of the distribution of vertical shearing stress along the central axis of symmetry of any cross-section (§ 12), and of the nature of the end-fixing which is permissible with the solution obtained in accordance with Saint-Venant's assumption.

So far the discussion in connection with the particular case of the circular sector cross-section has been made on the assumption of isotropy (and of uni-constant isotropy so far as the numerical work was concerned), but in §§ 15—19 there is given a projective method which leads to a series of aeolotropic solutions. The numerical work based on this brings out the important fact that aeolotropy such as occurs in a wooden beam (with the grain in the longitudinal direction) causes a great increase in the amount of the shearing torsion. It is found that the torsion of the similar isotropic beam has, very roughly, to be multiplied by $\frac{2}{5}E/\mu$, where E is the longitudinal stretch-modulus and μ a mean value of the corresponding transverse slide-modulus. For very high values of E/μ this rule ceases, however, to be even roughly true.

In the last part of the paper will be found brief discussions of some other cases of cross-sections which have been partially worked out. To begin with, an investigation is made of the algebraic solution of Saint-Venant. It is pointed out that he appears to have overlooked the torsional effect altogether in his discussion of the forms of uni-symmetrical cross-sections which arise from his solution. An attempt—with no great success—was made to extend his method to cases of asymmetry. Then follows a short description of a more general method of attacking the problem by aid of suitable conjugate functions. The case of the lemniscate is fully worked out. If there is not, as we fear there may be, some slip in the analysis, this section presents the interesting feature of a *reversed* torsional effect. Some details are given of a few other particular cases. We hope to be able to give a more adequate numerical discussion of these cases in a later paper.

* *Theorie der Elasticität fester Körper*, S. 74. Leipzig, 1862.

I. General Theory of the Flexure Problem.

§ 2. *General expressions for the displacements in an aeolotropic material where there are no mutual stresses between the 'fibres' perpendicular to those fibres.*

Let us take this longitudinal direction to be the axis of z and the axes of x and y to be at right angles to it. We shall denote the displacements at x, y, z by u, v, w and the stresses by $\widehat{xx}, \widehat{yy}, \widehat{zz}, \widehat{yz}, \widehat{zx}, \widehat{xy}$. Assuming uniform aeolotropy, and that the planes of the coordinates are planes of symmetry of structure, we shall use E to denote the stretch-modulus in the direction of the axis of z, μ_1, μ_2 to denote the slide-moduli in the planes of xz and yz respectively, and η_1, η_2 the Poisson's ratios for contraction in the directions of x and y due to an extension in the direction of z*.

The condition that the stresses perpendicular to the longitudinal axis (z) of the beam across planes parallel to that axis should be everywhere zero may be written analytically as

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0$$

everywhere, giving immediately

$$\left.\begin{aligned} -\frac{1}{\eta_1}\frac{\partial u}{\partial x} = -\frac{1}{\eta_2}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{aligned}\right\} \quad \text{.............................(1).}$$

The body-stress equations, also, reduce to

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \\ \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \\ \mu_1 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \mu_2 \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + E\frac{\partial^2 w}{\partial z^2} = 0 \end{aligned}\right\} \quad \text{...............(2).}$$

Fixing our attention on w to begin with, we deduce

$$\frac{\partial^3 w}{\partial z^3} = \frac{\partial^3 w}{\partial x^2 \partial z} = \frac{\partial^3 w}{\partial y^2 \partial z} = \frac{\partial^3 w}{\partial x \partial y \partial z} = 0 \quad \text{.........................(3),}$$

so that $\dfrac{\partial w}{\partial z}$, the longitudinal stretch, is linear in x, y and in z, and may be written

$$\frac{\partial w}{\partial z} = (A'x + B'y + C') - (Ax + By + C)z \quad \text{..................(4).}$$

* The arrangement of the subscripts in the notation is made different from the usual arrangement, because so far as the aeolotropy is concerned the problem is essentially two-dimensional.

Integrating the first set of (1) with this value of $\frac{\partial w}{\partial z}$, we get

$$\left.\begin{aligned} u &= -\eta_1(\tfrac{1}{2}A'x^2+B'xy+C'x)+\eta_1(\tfrac{1}{2}Ax^2+Bxy+Cx)z+F(y,z) \\ v &= -\eta_2(A'xy+\tfrac{1}{2}B'y^2+C'y)+\eta_2(Axy+\tfrac{1}{2}By^2+Cy)z+G(z,x) \\ w &= (A'x+B'y+C')z-\tfrac{1}{2}(Ax+By+C)z^2+H(x,y) \end{aligned}\right\}\ldots(5),$$

where F, G, H are functions to be determined to satisfy the remaining equations of (1) and (2).

Finally we arrive at the following as our displacements,

$$\left.\begin{aligned} u &= -\tau yz-\{\tfrac{1}{2}(\eta_1x^2-\eta_2y^2)A'+\eta_1xyB'+\eta_1xC'\}+\{\tfrac{1}{2}(\eta_1x^2-\eta_2y^2)A+\eta_1xyB+\eta_1xC\}z \\ &\quad -\tfrac{1}{2}A'z^2+\tfrac{1}{6}Az^3-\gamma y+\beta z+\alpha' \\ v &= \tau xz-\{\eta_2xyA'+\tfrac{1}{2}(\eta_2y^2-\eta_1x^2)B'+\eta_2yC'\}+\{\eta_2xyA+\tfrac{1}{2}(\eta_2y^2-\eta_1x^2)B+\eta_2yC\}z \\ &\quad -\tfrac{1}{2}B'z^2+\tfrac{1}{6}Bz^3-\alpha z+\gamma x+\beta' \\ w &= \chi(x,y)-(\mu_1\eta_1+\mu_2\eta_2-E)\left\{\frac{A}{2\mu_2}xy^2+\frac{B}{2\mu_1}x^2y+\frac{C}{4}\left(\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}\right)\right\} \\ &\quad +(A'x+B'y+C')z-\tfrac{1}{2}(Ax+By+C)z^2-\beta''x+\alpha''y+\gamma' \end{aligned}\right\}\ldots\ldots\ldots\ldots(6),$$

where τ and α, β, γ, α', β', γ', α'', β'' are constants of integration, and χ is a function of x and y which satisfies the equation

$$\mu_1\frac{\partial^2\chi}{\partial x^2}+\mu_2\frac{\partial^2\chi}{\partial y^2}=0 \ldots\ldots\ldots\ldots(7).$$

These equations may be tested by substitution in the 'equations of compatibility.' They are a generalisation of the equations given by Clebsch* for the isotropic case. The constants α'', β'', γ might of course be included under the arbitrary function χ, but it is clearer from the physical point of view to keep them separate since the linear terms of the displacements are those which must be used to fix the end-conditions. The terms in γ give rise to a rotation about the axis of z, and the other notation is arranged to suit the other possible rotations which would be given by $\alpha=\alpha''$ and $\beta=\beta''$.

The equations (6) and (7) give the most general displacements possible in a material which may be *supposed* to consist of parallel fibres *when there is no mutual stresses between the 'fibres' in any direction at right angles to the fibrous direction.* This was the assumption first made by Saint-Venant in his treatment of the flexure of a uniform prism by a load applied at its free end. It is an assumption justified by the consideration that the stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ must all be zero at the lateral surfaces of the prism, and that if they are taken to be zero throughout the interior of the prism we can deduce a solution of the flexure problem which can be made to satisfy certain end-conditions of fixing and of load. It is true that in this matter of fixing and loading we do not presume that the distribution of stresses over the cross-

* *Théorie de l'Elasticité:* Saint-Venant's annotated French translation (1883), p. 150.

sections at the ends is actually attained in the theory as the experience of practice would lead us to desire, and that we have to make use of the principle of the elastic equivalence of equipollent systems of loads, but Saint-Venant on physical grounds, and Boussinesq by analysis, have shown so conclusively the rapidity with which the local effects of external forces diminish with distance from the point of application that in any beam having a length several times greater than its breadth dimensions the principle may be freely applied.

We proceed now to the particular case of the prismatic beam under flexure, and after that to the particular prismatic beam which has for cross-section the sector of a circle or annulus.

§ 3. *Particular case of the foregoing theory for the cylindrical beam under flexure.*

The appropriate direction for the axis of z in the case of a right prism will of course be that parallel to the generating line. We will take the end that is fixed* to be $z=0$, and if the length of the beam is l the free end will be $z=l$. We assume a load to act on this free end in the direction of the axis of x, and such that it is statically equivalent to a force W acting through the centroid of the end cross-section.

Since the load acts at right angles to the longitudinal axis, the stress $\widehat{zz}$ will be zero all over the surface $z=l$, that is

$$\left(\frac{\partial w}{\partial z}\right)_{z=l} = (A'x + B'y + C') - (Ax + By + C)\,l = 0,$$

giving $$A' = Al, \qquad B' = Bl, \qquad C' = Cl,$$

and therefore $$\frac{\partial w}{\partial z} = (Ax + By + C)(l - z) \quad \text{.........................(8)}.$$

Now let us consider the conditions affecting the resultant traction over *any* cross-section of the beam. These conditions are obtained by forming the equations of equilibrium of the portion of the beam between the cross-section in question and the end $z=l$, the only forces being the load W and the tractions over the cross-section. The equations are

$$\left.\begin{array}{l} W = \iint \widehat{zx}\,dx\,dy, \quad 0 = \iint \widehat{yz}\,dx\,dy, \quad 0 = \iint \widehat{zz}\,dx\,dy, \\ -W\bar{y} = \iint (x\,.\,\widehat{yz} - y\,.\,\widehat{xz})\,dx\,dy, \quad -W(l-z) = \iint x\,.\,\widehat{zz}\,dx\,dy, \quad 0 = \iint y\,.\,\widehat{zz}\,dx\,dy \end{array}\right\} \quad \text{.......................(9)},$$

$\bar{y}$ being the y-coordinate of the centroid of the cross-section and the integrations taking place over the area of the cross-section.

* The exact mode of fixing the end of the beam and of applying the load at the other end will be discussed later (§ 14).

Consider first the integrals concerning $\widehat{zz}$.

From $0=\iint \widehat{zz}\,dxdy$ and equation (8) we derive

$$0=\iint (Ax+By+C)\,dxdy,$$

or
$$0=\iint (A(x-\bar{x})+B(y-\bar{y})+A\bar{x}+B\bar{y}+C)\,dxdy,$$

where $(\bar{x},\bar{y})$ is the centroid of the cross-section, and for this equation to be satisfied we must have

$$A\bar{x}+B\bar{y}+C=0 \quad\text{..............................(10).}$$

From
$$0=\iint y\,.\,\widehat{zz}\,dxdy,$$

or
$$0=\iint (Axy+By^2+Cy)\,dxdy,$$

we have

$$0=\iint (A(x-\bar{x})(y-\bar{y})+B(y-\bar{y})^2+A\bar{x}(y-\bar{y})+A\bar{y}(x-\bar{x})+2B\bar{y}(y-\bar{y})$$
$$+A\bar{x}\bar{y}+B\bar{y}^2+C\bar{y})\,dxdy,$$

and, using (10) and the property of the centroid,

$$0=\iint (A(x-\bar{x})(y-\bar{y})+B(y-\bar{y})^2)\,dxdy.$$

If* therefore we take the axes of x and y to be parallel to the principal axes of the section through the centroid and thus make the product moment

$$\iint (x-\bar{x})(y-\bar{y})\,dxdy$$

vanish, we must have

$$B=0 \quad\text{..................................(11).}$$

Again from

$$-W(l-z)=\iint x\,.\,\widehat{zz}\,dxdy$$

and equation (8) we have

$$-W(l-z)=E\iint (Ax^2+Bxy+Cx)(l-z)\,dxdy,$$

and using (10) and (11)

$$-W=E\iint A(x-\bar{x})^2\,dxdy,$$

or
$$A=-\frac{W}{EI} \quad\text{..(12),}$$

where
$$I=\iint (x-\bar{x})^2\,dxdy,$$

the second moment of the cross-section about the line through the centroid perpendicular to the plane of flexure.

* It is of course not *necessary* to make this simplification.

With these simplifications the longitudinal stretch is now given by

$$\frac{\partial w}{\partial z} = -\frac{W}{EI}(x-\bar{x})(l-z),$$

or, if we select as origin some point on that principal axis through the centroid which is at right angles to the plane of flexure, so that $\bar{x}=0$,

$$\frac{\partial w}{\partial z} = -\frac{W}{EI}x(l-z) \quad \text{.............................(13).}$$

An examination of the values of u, v, w in (6) will show that such a change of origin will be absorbed by the arbitrary constants and the function χ.

The expressions for the displacements now become

$$\left.\begin{aligned} u &= -\tau yz + \frac{W}{EI}\left[\tfrac{1}{2}(\eta_1 x^2 - \eta_2 y^2)(l-z) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right] - \gamma y + \beta z + \alpha' \\ v &= \tau zx + \frac{W}{EI}\eta_2 xy(l-z) - \alpha z + \gamma x + \beta' \\ w &= \chi(x, y) - \frac{W}{EI}\left[\left(\frac{E - \mu_1\eta_1 - \mu_2\eta_2}{2\mu_2}\right)xy^2 + x\left(lz - \tfrac{1}{2}z^2\right)\right] - \beta'' x + \alpha'' y + \gamma' \end{aligned}\right\} \text{...(14),}$$

where $\chi(x, y)$ satisfies

$$\mu_1 \frac{\partial^2 \chi}{\partial x^2} + \mu_2 \frac{\partial^2 \chi}{\partial y^2} = 0 \quad \text{.............................(15).}$$

To apply the remaining traction conditions of (9) we require expressions for the shears $\widehat{xz}$ and $\widehat{yz}$. These are

$$\left.\begin{aligned} \widehat{xz} &= \mu_1\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \mu_1\left\{-\tau y + \frac{\partial \chi}{\partial x} - \frac{W}{EI}\left[\tfrac{1}{2}\eta_1 x^2 + \tfrac{1}{2}\left(\frac{E - \eta_1\mu_1 - 2\eta_2\mu_2}{\mu_2}\right)y^2\right] + \beta - \beta''\right\} \\ \widehat{yz} &= \mu_2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \mu_2\left\{\tau x + \frac{\partial \chi}{\partial y} - \frac{W}{EI}\left(\frac{E - \eta_1\mu_1}{\mu_2}\right)xy - \alpha + \alpha''\right\} \end{aligned}\right\}$$

$$\text{.......................(16).}$$

The first of (9) may be written

$$W = \iint\left(\widehat{zx} + \left(\frac{\partial \widehat{zx}}{\partial x} + \frac{\partial \widehat{zy}}{\partial y} + \frac{Wx}{I}\right)x\right)dx\,dy$$

from (13) and the third body-stress equation of (2),

or
$$W = W + \iint\left(\frac{\partial}{\partial x}(x\,.\,\widehat{zx}) + \frac{\partial}{\partial y}(x\,.\,\widehat{yz})\right)dx\,dy$$

$$= W + \int x\left(\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu)\right)ds$$

by Green's theorem, ν being the normal outwards, the integration being round the boundary of the cross-section, and in order that this may be true we must have

$$\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu) = 0 \text{..........................(17)}$$

all over the boundary. We shall refer to this as the *Boundary Condition*.

Again substituting from (16) in

$$0 = \iint \widehat{yz}\, dx\, dy$$

we get
$$0 = \iint \left(\tau x + \frac{\partial \chi}{\partial y} - \frac{W}{EI} \left(\frac{E - \eta_1 \mu_1}{\mu_2} \right) xy - \alpha + \alpha'' \right) dx\, dy.$$

The integral $\iint x\, dx\, dy$ vanishes because of our choice of origin, and $\iint xy\, dx\, dy$ will vanish* *if the cross-section be symmetrical about the axis of y.* This is true in all the cases treated in this paper, and we will make this additional assumption in all that follows. We thus have

$$0 = \iint \left(\frac{\partial \chi}{\partial y} - \alpha + \alpha'' \right) dx\, dy,$$

and, transforming by Green's theorem into an integral round the boundary of the cross-section, we get

$$0 = \int (\chi - (\alpha - \alpha'')\, y) \cos (\nu y)\, ds \quad \text{.....................(18)},$$

an equation which we shall refer to as *Condition I.*

From the last remaining of the traction conditions (9)

$$-W\bar{y} = \iint (x \,.\, \widehat{yz} - y \,.\, \widehat{xz})\, dx\, dy,$$

we derive

$$\begin{aligned} -W\bar{y} = \iint \Big[&\tau (\mu_2 x^2 + \mu_1 y^2) + \left(\mu_2 x \frac{\partial \chi}{\partial y} - \mu_1 y \frac{\partial \chi}{\partial x} \right) - \frac{W}{EI} (E - \tfrac{3}{2} \eta_1 \mu_1)\, x^2 y \\ &+ \frac{W}{EI} \frac{(E - \eta_1 \mu_1 - 2\eta_2 \mu_2)\, \mu_1}{2\mu_2} y^3 - \mu_2 (\alpha - \alpha'')\, x - \mu_1 (\beta - \beta'')\, y \Big]\, dx\, dy \text{......(19)}. \end{aligned}$$

This we will refer to as *Condition II.*

We have now reduced the physical problem to the mathematical problem of finding the function $\chi(x, y)$ which will satisfy the differential equation

$$\mu_1 \frac{\partial^2 \chi}{\partial x^2} + \mu_2 \frac{\partial^2 \chi}{\partial y^2} = 0,$$

and at the same time satisfy the three conditions given in (17), (18), (19) for the particular cross-section we have in mind. This at least is the ideal problem; in reality the more immediate problem is to find the particular cross-section which will possess as nearly as possible the special characteristics we want to investigate and which will at the same time permit of a suitable simple function χ being found for it.

In addition to our original assumption of no mutual stress between the

* The integral will also vanish if the origin be taken at the centroid but soluble cases will then usually be symmetrical about *both* principal axes.

longitudinal 'fibres' of the beam perpendicular to their direction, it is to be remembered that we have also assumed

1°, that the plane of flexure is parallel to a principal plane of the beam through the centroidal line, and

2°, that the beam is symmetrical about the plane through the centroidal line and perpendicular to the plane of flexure, an assumption which makes 1° immediately satisfied.

§ 4. *The case of isotropy.*

It will be convenient at this stage to postpone the aeolotropic discussion and to summarise the foregoing for the isotropic case where $\eta_1 = \eta_2 = \eta$ and $\mu_1 = \mu_2 = \mu$.

We have taken the longitudinal axis of the beam to be the axis of z and the plane yz to be a plane of symmetry, the flexion taking place in the plane zx. The fixed end of the beam is at $z = 0$ and the loaded end at $z = l$.

The displacements* are

$$\left.\begin{aligned} u &= -\tau yz + \frac{W}{EI}\left[\tfrac{1}{2}\eta(x^2 - y^2)(l - z) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right] - \gamma y + \beta z + \alpha' \\ v &= \tau zx + \frac{W}{EI}\eta xy(l - z) + \gamma x - \alpha z + \beta' \\ w &= \chi(x, y) - \frac{W}{EI}\left[x\left(lz - \tfrac{1}{2}z^2\right) + xy^2\right] - \beta'' x + \alpha'' y + \gamma' \end{aligned}\right\} \quad \text{......(20)},$$

where

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = 0 \quad \text{..............................(21)},$$

and τ, α, β, γ, α', β', γ', α'', β'' are arbitrary constants.

In addition we have the following conditions to be satisfied.

(i) The Boundary Condition:

$$\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu) = 0 \quad \text{.........................(22)},$$

where

$$\widehat{xz} = \mu\left\{-\tau y + \frac{\partial \chi}{\partial x} - \frac{W}{EI}\left(\tfrac{1}{2}\eta x^2 + \left(1 - \tfrac{1}{2}\eta\right)y^2\right) + \beta - \beta''\right\}$$

and

$$\widehat{yz} = \mu\left\{\tau x + \frac{\partial \chi}{\partial y} - \frac{W}{EI}(2 + \eta)xy - \alpha + \alpha''\right\}.$$

(ii) Condition I:

$$0 = \int (\chi - (\alpha - \alpha'')y)\cos(y\nu)\,ds \quad \text{.....................(23)},$$

the integral being round the boundary of any cross-section, and

(iii) Condition II:

$$-W\bar{y} = \mu\iint\left[\tau(x^2 + y^2) + \left(x\frac{\partial \chi}{\partial y} - y\frac{\partial \chi}{\partial x}\right) - \frac{W}{EI}\left(2 + \tfrac{1}{2}\eta\right)x^2 y + \frac{W}{EI}\left(1 - \tfrac{1}{2}\eta\right)y^3 - (\beta - \beta'')y\right]dx\,dy \quad \text{...........(24)}.$$

* Compare Love's *Elasticity*, 2nd Edition, Chapter xv, Equations (12), p. 319. The axes of coordinates and notation used in the present paper make comparison easy with the results given in Chapter xv.

II. Case of a Beam having as Cross-section a Circular or Annular Sector.

§ 5. *The equations suitable to the case of the circular sector cross-section with isotropy.*

For our problem of the sector-shaped cross-section we must now transform our equations to cylindrical coordinates z, r, θ. Let O, the centre of the circular arcs be our origin, and let the axis of y be the line $\theta = 0$. We shall take the sector to be bounded by the arcs $r = a$ and $r = a_0$, and the lines $\theta = \pm\frac{1}{2}\gamma$.

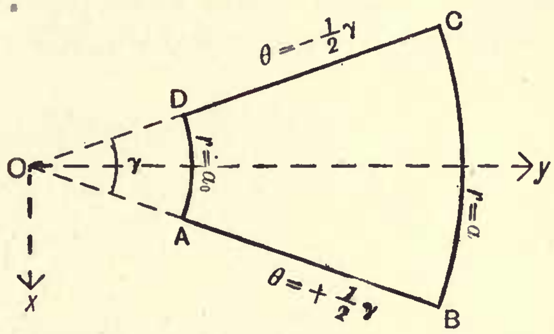


Fig. 1.

The boundary condition

$$\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu) = 0$$

is, on substitution of $\widehat{xz}$ and $\widehat{yz}$,

$$\frac{\partial\chi}{\partial x}\cos(x\nu) + \frac{\partial\chi}{\partial y}\cos(y\nu)$$

$$= \tau(y\cos(x\nu) - x\cos(y\nu)) + \frac{W}{EI}[(\tfrac{1}{2}\eta x^2 + (1-\tfrac{1}{2}\eta)y^2)\cos(x\nu) + (2+\eta)xy\cos(y\nu)]$$
$$-(\beta-\beta'')\cos(x\nu) + (\alpha-\alpha'')\cos(y\nu) \quad\ldots\ldots\ldots\ldots(25),$$

and we can now obtain the conditions that are to be satisfied on each of the edges of the sector.

1°. Over the external arc, where $r = a$ and $-\frac{1}{2}\gamma \leqslant \theta \leqslant +\frac{1}{2}\gamma$, the boundary condition becomes

$$\left(\frac{\partial\chi}{\partial r}\right)_{r=a} = \tau(a\cos\theta\sin\theta - a\sin\theta\cos\theta) + \frac{W}{EI}[(\tfrac{1}{2}\eta a^2\sin^2\theta + (1-\tfrac{1}{2}\eta)a^2\cos^2\theta)\sin\theta$$
$$+ (2+\eta)a^2\sin\theta\cos^2\theta] - (\beta-\beta'')\sin\theta + (\alpha-\alpha'')\cos\theta,$$

or

$$\left(\frac{\partial\chi}{\partial r}\right)_{r=a} = \frac{Wa^2}{EI}[(\tfrac{3}{4}+\tfrac{1}{2}\eta)\sin\theta + \tfrac{3}{4}\sin 3\theta] - (\beta-\beta'')\sin\theta + (\alpha-\alpha'')\cos\theta \ldots\ldots(26).$$

2°. Over the internal arc, it is easily found that

$$\left(\frac{\partial\chi}{\partial r}\right)_{r=a_0} = \frac{Wa_0^2}{EI}[(\tfrac{3}{4}+\tfrac{1}{2}\eta)\sin\theta + \tfrac{3}{4}\sin 3\theta] - (\beta-\beta'')\sin\theta + (\alpha-\alpha'')\cos\theta \ldots\ldots(26)'.$$

3°. Over the radial edge, $a_0 \leqslant r \leqslant a$, and $\theta = \frac{1}{2}\gamma$, the condition is, in the same way,

$$\left(\frac{\partial\chi}{r\partial\theta}\right)_{\theta=\frac{1}{2}\gamma} = \tau r + \frac{Wr^2}{EI}[(\tfrac{1}{4}-\tfrac{1}{2}\eta)\cos\tfrac{1}{2}\gamma + \tfrac{3}{4}\cos\tfrac{3}{2}\gamma]$$
$$-(\beta-\beta'')\cos\tfrac{1}{2}\gamma - (\alpha-\alpha'')\sin\tfrac{1}{2}\gamma \ldots\ldots(27),$$

and, 4°, over the other radial edge $\theta = -\frac{1}{2}\gamma$,

$$\left(\frac{\partial\chi}{r\partial\theta}\right)_{\theta=-\frac{1}{2}\gamma} = \tau r + \frac{Wr^2}{EI}\left[\left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\cos\tfrac{1}{2}\gamma + \tfrac{3}{4}\cos\tfrac{3}{2}\gamma\right] - (\beta - \beta'')\cos\tfrac{1}{2}\gamma + (\alpha - \alpha'')\sin\tfrac{1}{2}\gamma \quad\ldots\ldots(27)',$$

the only difference between 3° and 4° being the sign of the term containing $\alpha - \alpha''$. But if χ is to be one-valued, the conditions 3° and 4° cannot both be satisfied unless $\alpha = \alpha''$. It will be evident after we have found χ that the physical interpretation of this is that, in a beam which is symmetrical about the plane of yz, there is *no lateral slide* when a flexing load is applied in the principal plane zx.

§ 6. *The solution for χ suitable to the case of the sector.*

We shall now assume*, as our solution of

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 0,$$

$$\chi = \sum_m \left(A_m r^m + \frac{B_m}{r^m}\right)\sin m\theta + a_1 r\sin\theta + a_2 r^2 \sin 2\theta + a_3 r^3 \sin 3\theta \quad\ldots(28),$$

and proceed to determine a_1, a_2, a_3, A_1, A_2, ..., B_1, B_2, ... and the appropriate values of m from the conditions obtained above.

From the boundary condition over the straight edges

$$\left(\frac{\partial\chi}{r\partial\theta}\right)_{\theta=\frac{1}{2}\gamma} \equiv \Sigma\left(A_m r^{m-1} + \frac{B_m}{r^{m+1}}\right) m\cos\tfrac{1}{2}m\gamma + a_1\cos\tfrac{1}{2}\gamma + a_2 r \,.\, 2\cos\gamma + a_3 r^2 \,.\, 3\cos\tfrac{3}{2}\gamma$$

$$= \tau r + \frac{Wr^2}{EI}\left[\left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\cos\tfrac{1}{2}\gamma + \tfrac{3}{4}\cos\tfrac{3}{2}\gamma\right] - (\beta - \beta'')\cos\tfrac{1}{2}\gamma \quad\ldots\ldots(29),$$

which will be satisfied only if

$$\frac{m\gamma}{2} = (2i+1)\frac{\pi}{2},$$

where i is zero or a positive integer,

and $\quad a_1 = -(\beta - \beta''), \qquad a_2 = \dfrac{\tau}{2\cos\gamma}, \qquad a_3 = \dfrac{1}{3}\dfrac{W}{EI}\left[\tfrac{3}{4} + \left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\dfrac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right].$

Hence we have

$$\chi = \sum_{i=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^m}\right)\sin m\theta - (\beta - \beta'')r\sin\theta + \frac{\tau}{2\cos\gamma}r^2\sin 2\theta + \frac{W}{3EI}\left[\tfrac{3}{4} + \left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right] r^3 \sin 3\theta \quad\ldots\ldots\ldots(30).$$

* We have to make χ an *odd* function of θ to satisfy the equations of § 5. Physically this means that the longitudinal displacements in the beam are symmetrical, with change of sign, about the axis of symmetry of the cross-section.

Applying now the boundary condition over the arcs, we have

$$\left(\frac{\partial\chi}{\partial r}\right)_{r=a} = \sum_{i=0}^{\infty}\left(A_m m a^{m-1} - m\frac{B_m}{a^{m+1}}\right)\sin m\theta$$

$$-(\beta-\beta'')\sin\theta + \frac{\tau}{\cos\gamma}a\sin 2\theta + \frac{W}{EI}\left[\tfrac{3}{4}+(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right]a^2\sin 3\theta$$

$$= \frac{Wa^2}{EI}\left[(\tfrac{3}{4}+\tfrac{1}{2}\eta)\sin\theta + \tfrac{3}{4}\sin 3\theta\right] - (\beta-\beta'')\sin\theta,$$

or

$$\sum_{i=0}^{\infty}\left(A_m a^{m-1} - \frac{B_m}{a^{m+1}}\right) m\sin m\theta$$

$$= \frac{Wa^2}{EI}(\tfrac{3}{4}+\tfrac{1}{2}\eta)\sin\theta - \frac{\tau a}{\cos\gamma}\sin 2\theta - \frac{Wa^2}{EI}(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\sin 3\theta \quad\text{...(31)},$$

along with a similar equation in a_0 for the internal surface.

These we may write for brevity as

$$\sum_{i=0}^{\infty} C_m \sin m\theta = \sum_p c_p \sin p\theta \qquad [p=1, 2, 3] \quad\text{...............(32)},$$

where c_p is known and C_m contains the quantities A_m and B_m which are to be found

We can now determine the constants C_m by Fourier's method and thereafter obtain A_m and B_m. Multiply both sides of the equation (32) by $\sin m\theta$ and integrate between the limits $-\frac{1}{2}\gamma$ and $\frac{1}{2}\gamma$. Then

$$C_m\int_{-\frac{1}{2}\gamma}^{\frac{1}{2}\gamma}\sin^2 m\theta\,d\theta = \sum_p c_p\int_{-\frac{1}{2}\gamma}^{\frac{1}{2}\gamma}\sin p\theta\sin m\theta\,d\theta$$

$$= \tfrac{1}{2}\sum_p c_p\int_{-\frac{1}{2}\gamma}^{\frac{1}{2}\gamma}\{\cos(p-m)\theta - \cos(p+m)\theta\}\,d\theta,$$

and

$$C_m = \frac{1}{\gamma}\sum_p 2c_p\left(\frac{\sin\frac{1}{2}(p-m)\gamma}{p-m} - \frac{\sin\frac{1}{2}(p+m)\gamma}{p+m}\right)$$

$$= \frac{1}{\gamma}\sum_p\frac{4c_p p\cos\frac{1}{2}p\gamma}{m^2-p^2}(-1)^i \quad\text{...................................(33)}.$$

Replacing the notation C_m and c_p, we have, after a little reduction,

$$mA_m a^{m-1} - m\frac{B_m}{a^{m+1}} = \frac{W}{EI}\frac{3+2\eta}{m^2-1}\frac{\cos\frac{1}{2}\gamma}{\gamma}(-1)^i a^2 - \frac{8\tau(-1)^i}{(m^2-4)\gamma}a$$

$$-\frac{3W}{EI}\frac{1-2\eta}{m^2-9}\frac{\cos\frac{1}{2}\gamma}{\gamma}(-1)^i a^2,$$

$$mA_m a_0^{m-1} - m\frac{B_m}{a_0^{m+1}} = \frac{W}{EI}\frac{3+2\eta}{m^2-1}\frac{\cos\frac{1}{2}\gamma}{\gamma}(-1)^i a_0^2 - \frac{8\tau(-1)^i}{(m^2-4)\gamma}a_0$$

$$-\frac{3W}{EI}\frac{1-2\eta}{m^2-9}\frac{\cos\frac{1}{2}\gamma}{\gamma}(-1)^i a_0^2,$$

the solution of which is

$$\left.\begin{aligned}
A_m &= -\frac{8\tau(-1)^i}{(m^2-4)\,m\gamma}\,\frac{a^{m+2}-a_0^{\,m+2}}{a^{2m}-a_0^{\,2m}} \\
&\qquad +(-1)^i\,\frac{W}{EI}\,\frac{\cos\frac{1}{2}\gamma}{m\gamma}\left(\frac{3+2\eta}{m^2-1}-\frac{3(1-2\eta)}{m^2-9}\right)\frac{a^{m+3}-a_0^{\,m+3}}{a^{2m}-a_0^{\,2m}} \\
B_m &= \frac{8\tau(-1)^i}{(m^2-4)\,m\gamma}\,\frac{a^{m+2}a_0^{\,m+2}\left(a^{m-2}-a_0^{\,m-2}\right)}{a^{2m}-a_0^{\,2m}} \\
&\qquad -(-1)^i\,\frac{W}{EI}\,\frac{\cos\frac{1}{2}\gamma}{m\gamma}\left(\frac{3+2\eta}{m^2-1}-\frac{3(1-2\eta)}{m^2-9}\right)\frac{a^{m+3}a_0^{\,m+3}\left(a^{m-3}-a_0^{\,m-3}\right)}{a^{2m}-a_0^{\,2m}}
\end{aligned}\right\}\ \ldots(34).$$

We thus have χ and therefore the displacements u, v, w expressed in terms of τ and the arbitrary constants α, β, β'', γ, α', β', γ':

$$\left.\begin{aligned}
u &= -\tau yz+\frac{W}{EI}\left[\tfrac{1}{2}(\eta x^2-\eta y^2)(l-z)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right]-\gamma y+\beta z+\alpha' \\
v &= \tau zx+\frac{W}{EI}\eta xy(l-z)-\alpha z+\gamma x+\beta' \\
w &= \Bigg[-\sum_{i=0}^{\infty}\frac{8\tau(-1)^i}{(m^2-4)\,m\gamma}\,\frac{a^{m+2}-a_0^{\,m+2}}{a^{2m}-a_0^{\,2m}}\,r^m+\sum_{i=0}^{\infty}\frac{8\tau(-1)^i}{(m^2-4)\,m\gamma}\,\frac{a^{m+2}a_0^{\,m+2}\left(a^{m-2}-a_0^{\,m-2}\right)}{a^{2m}-a_0^{\,2m}}\,\frac{1}{r^m} \\
&\quad +\sum_{i=0}^{\infty}(-1)^i\,\frac{W}{EI}\,\frac{\cos\frac{1}{2}\gamma}{m\gamma}\left(\frac{3+2\eta}{m^2-1}-\frac{3(1-2\eta)}{m^2-9}\right)\frac{a^{m+3}-a_0^{\,m+3}}{a^{2m}-a_0^{\,2m}}\,r^m \\
&\quad -\sum_{i=0}^{\infty}(-1)^i\,\frac{W}{EI}\,\frac{\cos\frac{1}{2}\gamma}{m\gamma}\left(\frac{3+2\eta}{m^2-1}-\frac{3(1-2\eta)}{m^2-9}\right)\frac{a^{m+3}a_0^{\,m+3}\left(a^{m-3}-a_0^{\,m-3}\right)}{a^{2m}-a_0^{\,2m}}\,\frac{1}{r^m}\Bigg]\sin m\theta \\
&\quad -(\beta-\beta'')\,r\sin\theta+\frac{\tau}{2\cos\gamma}\,r^2\sin 2\theta+\frac{W}{3EI}\left[\tfrac{3}{4}+\left(\tfrac{1}{4}-\tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right]r^3\sin 3\theta \\
&\quad -\frac{W}{EI}\left[x\left(lz-\tfrac{1}{2}z^2\right)+xy^2\right]-\beta''x+\alpha''y+\gamma'
\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

where $$x=r\sin\theta,\quad y=r\cos\theta$$

and $$m=(2i+1)\frac{\pi}{\gamma}.$$

We note that the displacement in the direction of the flexing force, u, is symmetrical both in sign and magnitude about the radius of symmetry, while v and w have both alternating symmetry.

§ 7. *Application of the Conditions I and II and derivation of the value of the torsion* τ.

We have now to apply the Conditions I and II, equations (23) and (24) of § 4, and will thus determine two relations between τ and $\beta-\beta''$ from which these may be obtained.

(I) Condition I, that is

$$\int \chi \cos (y\nu)\, ds = 0 \quad \text{.............................(36)},$$

the integration being round the contour of any cross-section.

Taking the four parts AB, BC, CD, DA (see Fig. 1) separately, we have:

1°. Along AB, $\cos(y\nu) = -\sin\frac{1}{2}\gamma$, and

$$\int_A^B \chi \cos (y\nu)\, ds = -\sin\tfrac{1}{2}\gamma \int_{a_0}^a \chi_{\theta=\frac{1}{2}\gamma}\, dr.$$

2°. Along CD, in the same way,

$$\int_C^D \chi \cos (y\nu)\, ds = -\sin\tfrac{1}{2}\gamma \int_a^{a_0} \chi_{\theta=-\frac{1}{2}\gamma}\,(-dr) = -\int_A^B \chi \cos (y\nu)\, ds.$$

3°. Along BC, $\cos(y\nu) = \cos\theta$, but χ is odd about $\theta = 0$, and therefore the integral vanishes.

4°. Along DA, the integral again vanishes.

Thus Condition I is identically satisfied.

It is easy to see from (36) that Condition I is identically satisfied whenever the section has uniaxial symmetry about an axis perpendicular to the plane of loading. For in this case χ changes sign as we cross the axis, and, if P and P' be corresponding points on either side of the axis, an element ds at P will correspond to an element ds at P', $\cos(\nu y)$ will be the same at both points, but χ will change sign as containing only *odd* powers of x, and thus the two elements of the integral in (36) just cancel.

(II) Condition II, that is

$$-W\bar{y} = \mu \iint \Big[\tau (x^2 + y^2) + \left(x \frac{\partial \chi}{\partial y} - y \frac{\partial \chi}{\partial x} \right)$$
$$- \frac{W}{EI}(2 + \tfrac{1}{2}\eta)\, x^2 y + \frac{W}{EI}(1 - \tfrac{1}{2}\eta)\, y^3 - (\beta - \beta'')\, y \Big]\, dx\, dy.$$

When the integrations are carried out, we obtain

$$\mu \iint \left(x \frac{\partial \chi}{\partial y} - y \frac{\partial \chi}{\partial x} \right) dx\, dy = -\mu\tau\, \tfrac{1}{4}\gamma\, (a^4 - a_0^4) + (\beta - \beta'')\, \mu\, \tfrac{2}{3}\, (a^3 - a_0^3) \sin\tfrac{1}{2}\gamma$$
$$- W \frac{4}{3} \frac{a^3 - a_0^3}{a^2 - a_0^2} \frac{\sin\frac{1}{2}\gamma}{\gamma} - \frac{2}{5} \frac{W\mu}{EI} (a^5 - a_0^5) \left[(\tfrac{1}{4} - \tfrac{1}{2}\eta) \sin\tfrac{1}{2}\gamma + \tfrac{1}{4} \sin\tfrac{3}{2}\gamma \right] \text{...(37)},$$

and, transforming the surface integral by Green's theorem,

$$\iint \left(x \frac{\partial \chi}{\partial y} - y \frac{\partial \chi}{\partial x} \right) dx\, dy = \int \chi\, (x \cos (y\nu) - y \cos (x\nu))\, ds.$$

Taking the line integral in four parts, as in the previous case, we have:

1°. Along AB, $\quad \cos(y\nu) = -\sin\frac{1}{2}\gamma$, $\quad \cos(x\nu) = \cos\frac{1}{2}\gamma$, and

$$\int_A^B \chi\, (x \cos (y\nu) - y \cos (x\nu))\, ds = -\int_{a_0}^a \chi_{\theta=\frac{1}{2}\gamma}\, (r \sin^2\tfrac{1}{2}\gamma + r \cos^2\tfrac{1}{2}\gamma)\, dr = -\int_{a_0}^a \chi_{\theta=\frac{1}{2}\gamma}\, r\, dr.$$

2°. Along CD, in the same way,

$$\int_C^D \chi \,(x \cos(y\nu) - y \cos(x\nu))\, ds = + \int_{a_0}^{a} \chi_{\theta=\frac{1}{2}\gamma}\, r\,(-dr),$$

and therefore the sum of the values along AB and CD

$$= -2 \int_{a_0}^{a} \chi_{\theta=\frac{1}{2}\gamma}\, r dr.$$

3°. Along BC,

$$\int_B^C \chi \,(x \cos(y\nu) - y \cos(x\nu))\, ds = \int \chi \,(a \sin\theta \cos\theta - a \cos\theta \sin\theta)\, ds = 0,$$

and, 4°, along DA the integral again vanishes.

Thus the whole integral, P, is simply $-2 \int_{a_0}^{a} \chi_{\theta=\frac{1}{2}\gamma}\, r dr.$

Substituting χ from (30) we find:

$$\begin{aligned} P = -2 \Bigg[\Bigg\{ & \sum_{i=0}^{\infty} \frac{A_m}{m+2} (a^{m+2} - a_0^{m+2}) + \sum_{i=0}^{\infty} \frac{B_m}{m-2} \frac{a^{m-2} - a_0^{m-2}}{a^{m-2} a_0^{m-2}} \sin \tfrac{1}{2} m\gamma \Bigg\} \\ & - (\beta - \beta'')\, \tfrac{1}{3} (a^3 - a_0^3) \sin \tfrac{1}{2}\gamma + \tau \tfrac{1}{8} \tan\gamma\, (a^4 - a_0^4) \\ & + \frac{1}{15} \frac{W}{EI} \left(\tfrac{3}{4} + (\tfrac{1}{4} - \tfrac{1}{2}\eta) \frac{\cos \frac{1}{2}\gamma}{\cos \frac{3}{2}\gamma} \right) \sin \tfrac{3}{2}\gamma\, (a^5 - a_0^5) \Bigg] \quad \ldots(38), \end{aligned}$$

A_m and B_m being given by the formulae of equation (34), and on the substitution of this value of P in (37), of the values of A_m and B_m from (34) and the value of I we obtain, remembering that $E = 2\mu\,(1+\eta)$, after some reductions the following equation for τ:

$$\left.\begin{aligned} \tau \Bigg[& -(a^4 - a_0^4)(\tan\gamma - \gamma) + \sum_{i=0}^{\infty} \frac{64}{(m^2-4)(m+2)\, m\gamma} \frac{(a^{m+2} - a_0^{m+2})^2}{a^{2m} - a_0^{2m}} \\ & - \sum_{i=0}^{\infty} \frac{64}{(m^2-4)(m-2)\, m\gamma} \frac{a^4 a_0^4 (a^{m-2} - a_0^{m-2})^2}{a^{2m} - a_0^{2m}} \Bigg] \\ = \frac{W}{EI} \Bigg[& \sum_{i=0}^{\infty} \frac{2 \cos \frac{1}{2}\gamma}{(m+2)\, m\gamma} \left(\frac{4(3+2\eta)}{m^2-1} - \frac{12(1-2\eta)}{m^2-9} \right) \frac{(a^{m+3} - a_0^{m+3})(a^{m+2} - a_0^{m+2})}{a^{2m} - a_0^{2m}} \\ & - \sum_{i=0}^{\infty} \frac{2 \cos \frac{1}{2}\gamma}{(m-2)\, m\gamma} \left(\frac{4(3+2\eta)}{m^2-1} - \frac{12(1-2\eta)}{m^2-9} \right) \frac{a^5 a_0^5 (a^{m-3} - a_0^{m-3})(a^{m-2} - a_0^{m-2})}{a^{2m} - a_0^{2m}} \\ & + \tfrac{2}{15} (1-2\eta) \cos \tfrac{1}{2}\gamma \tan \tfrac{3}{2}\gamma\, (a^5 - a_0^5) \\ & - \tfrac{4}{3} (1+\eta) \left(1 - \frac{\sin\gamma}{\gamma}\right) \sin \tfrac{1}{2}\gamma \frac{(a^3 - a_0^3)(a^4 - a_0^4)}{a^2 - a_0^2} - \tfrac{2}{5} (1-2\eta) \sin \tfrac{1}{2}\gamma\, (a^5 - a_0^5) \Bigg] \end{aligned}\right\} \ldots(39).$$

The value of $\beta - \beta''$ still outstanding will flow from the nature of the terminal conditions.

As we shall see in § 9, it is this quantity τ which gives the torsion in the beam due to the asymmetrical distribution of the shears in any cross-section.

§ 8. *The solution in the case of the complete or non-curtate circular sector.*

For the case of the complete sector, that is when a_0 is taken to be zero, the solution of §§ 6 and 7 simplifies into the following:

$$\chi = \Sigma A_m r^m \sin m\theta - (\beta - \beta'') r \sin\theta + \frac{\tau}{2\cos\gamma} r^2 \sin 2\theta$$
$$+ \frac{W}{3EI}\left[\tfrac{3}{4} + (\tfrac{1}{4} - \tfrac{1}{2}\eta)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right] r^3 \sin 3\theta \text{......(40)},$$

where $A_m = -\dfrac{8\tau(-1)^i}{(m^2-4)m\gamma}\dfrac{1}{a^{m-2}} + \dfrac{(-1)^i W}{EI}\dfrac{\cos\frac{1}{2}\gamma}{m\gamma}\left(\dfrac{3+2\eta}{m^2-1} - \dfrac{3(1-2\eta)}{m^2-9}\right)\dfrac{1}{a^{m-3}}$

and $\qquad m = (2i+1)\dfrac{\pi}{\gamma}, \qquad i = 0, 1, 2, \ldots.$

and the torsion is given by

$$\tau\left[-\tan\gamma + \gamma + \frac{64}{\gamma}\sum_{i=0}^{\infty}\frac{1}{(m^2-4)(m+2)m}\right]$$
$$= \frac{Wa}{EI}\left[\sum_{i=0}^{\infty}\frac{2\cos\frac{1}{2}\gamma}{(m+2)m\gamma}\left(\frac{4(3+2\eta)}{m^2-1} - \frac{12(1-2\eta)}{m^2-9}\right)\right.$$
$$\left. + \tfrac{2}{15}(1-2\eta)\cos\tfrac{1}{2}\gamma\tan\tfrac{3}{2}\gamma - \tfrac{2}{5}(1-2\eta)\sin\tfrac{1}{2}\gamma - \tfrac{4}{3}(1+\eta)\left(1 - \frac{\sin\gamma}{\gamma}\right)\sin\tfrac{1}{2}\gamma\right] \text{..........................(41)}.$$

III. Numerical Work in connection with the Torsion due to Flexure. Discussion of Shear Distribution and of Terminal Conditions.

§ 9. *Torsion of beam with flexure.*

As stated at the beginning of the paper the torsion of the beam about its axis is due (i) to the resultant torsional couple which arises from the shears in directions at right angles to the axis and (ii) to the anticlastic curvature caused by the inequality of the longitudinal tension and consequent inequality of the Poisson contraction over any cross-section. For each of these effects to cause torsion it is of course essential that the cross-section be asymmetrical.

To determine the amount of torsion let us take the vertical displacements of points on the horizontal axis of symmetry at the fixed end and at the other end of the beam. These are given by the equation (20)

$$u = -\tau yz + \frac{W}{EI}\{\tfrac{1}{2}\eta(l-z)(x^2-y^2) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\} - \gamma y + \beta z + \alpha'.$$

Firstly, at the fixed end of the beam the displacement, (i) at the outermost point of the axis of symmetry, namely $(0, a, 0)$, is

$$u_a = -\frac{W}{EI}\tfrac{1}{2}\eta l a^2 - \gamma a + \alpha',$$

and (ii) at the inner end of the axis $(0, a_0, 0)$,

$$u_{a_0} = -\frac{W}{EI}\tfrac{1}{2}\eta l a_0^2 - \gamma a_0 + \alpha',$$

giving
$$u_a - u_{a_0} = -\frac{W}{EI}\tfrac{1}{2}\eta l (a^2 - a_0^2) - \gamma (a - a_0).$$

The first term arises through the anticlastic curvature and the second, being merely a rotational term affecting the whole beam, depends on the particular method of fixing the end of the beam. When we consider the torsion of one end relative to the other, it will cancel out.

At the end where the load is applied the corresponding displacements are

(i) at $(0, a, l)$
$$u_a' = -\tau a l + \frac{W}{EI}\tfrac{1}{3}l^3 - \gamma a + \beta l + \alpha',$$

and (ii) at $(0, a_0, l)$
$$u_{a_0}' = -\tau a_0 l + \frac{W}{EI}\tfrac{1}{3}l^3 - \gamma a_0 + \beta l + \alpha',$$

giving
$$u_a' - u_{a_0}' = -\tau l (a - a_0) - \gamma (a - a_0),$$

the terms $\frac{W}{EI}\tfrac{1}{3}l^3 + \beta l + \alpha'$ disappearing since they are really the expression for the flexure.

Taking
$$\psi_0{}^* = \frac{u_a - u_{a_0}}{a - a_0} = -\frac{W}{EI}\tfrac{1}{2}\eta l (a + a_0) - \gamma$$

and
$$\psi_l = \frac{u_a' - u_{a_0}'}{a - a_0} = -\tau l - \gamma,$$

as approximately giving the small angles of rotation at the two ends of the beam, we have the total torsion of the loaded end relative to the fixed end about the longitudinal axis of the beam

$$= \psi_l - \psi_0 = -\tau l + \frac{W}{EI}\tfrac{1}{2}\eta l (a + a_0) \quad \text{.........(42)},$$

the direction of the torsion $\psi_l - \psi_0$ being as shown in the diagram of the free end of the beam (Fig. 2).

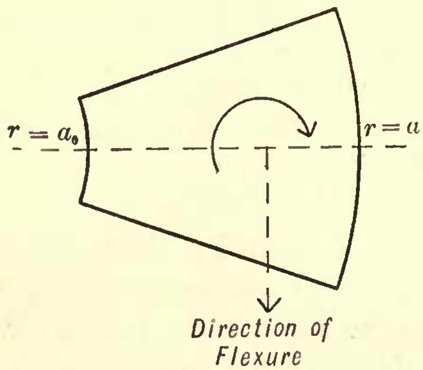


FIG. 2.

In connection with the anticlastic curvature it is to be noted that it is zero at the free end, there being no stretch there, and that, accordingly, there is no distortion of the surface at the free end except for the Saint-Venant distortion in the direction of the beam's longitudinal axis. Nevertheless if we were to clamp

* At first sight this expression for ψ_0 looks strange in that it would appear that the angle of torsion due to the anticlastic curvature would be infinitely large for the limiting case of a rectangular beam which is given by taking the curtate sectors to have infinite radii a and a_0 but such that $a - a_0$ remains finite. In taking the limit of $\frac{W}{EI}(a + a_0)$, however, it is to be remembered that we must take the limiting expression for I as well, and I being $\frac{1}{8}(a^4 - a_0^4)(\gamma - \sin\gamma)$ it is seen that the angle of torsion is proportional to $\frac{1}{(a^2 + a_0^2)(a + a_0)}$ which gives zero angle of torsion for the rectangular beam as it should do.

the fixed end in such a way that the ends of the axis of symmetry $(0, a, 0)$ and $(0, a_0, 0)$ were fixed, the anticlastic curvature of the beam at that end would give rise to the torsion $\frac{W}{EI}\frac{1}{2}\eta l\,(a+a_0)$ at the free end. If the fixed end be clamped in another way, the visible torsion due to the anticlastic curvature would be less and, it is clear, might even be zero. This would happen, for example, if the clamping were such that it fixed that element of the axis of symmetry, $x=0$, which, after the curving due to the flexure, remained in a direction at right angles to the plane of flexure. In the practical case of the beam which approximates both in length* and cross-section to the shape of the section of a propeller-blade, it is found that the anticlastic curvature gives rise to a rotation about half that due to the shear torsion but *opposite* in sign.

It must be borne in mind that the anticlastic torsion depends on the nature of the 'fixing' and the methods of clamping used in practice differ very widely from the theoretical means of fixing provided by the constants α, β, γ, α', β', γ', α'', β'' of our displacement equations. These constants occur only in the linear parts of the equations so that the most we can do with them is to fix certain points and certain coefficients of the scheme $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at points on the plane $z=0$. Remembering that we have found that $\alpha=\alpha''$, and that there is an equation for $(\beta-\beta'')$, we see how easy it is to exhaust the stock of arbitrary constants. Thus, if we try to fix an element at the point $(x, y, 0)$, we find that making $u=v=w=0$ and $\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}=\frac{\partial u}{\partial z}\left(\equiv\frac{\partial v}{\partial x}\right)=0$ gives all the arbitrary constants definite values. Similar difficulties arise with the method of loading, where the assumptions of the theory are even more difficult to approximate to in an actual experiment.

§ 10. *The torsion of the non-curtate sector for various values of the angle* γ.

1°. γ very small.

To get an approximate value of τ when γ is small we can expand the trigonometrical functions and the three series of (41) in powers of γ.

If
$$S_n=\sum_{i=0}^{\infty}\frac{1}{(m+2)\,m\gamma\,(m^2-n^2)}\qquad[n=1, 2, 3],$$

then
$$S_n=\sum_{i=0}^{\infty}\frac{1}{m^4\gamma\left(1+\frac{2}{m}\right)\left(1-\frac{n^2}{m^2}\right)}$$
$$=\sum_{p=2}^{p=\infty}\left\{{}_nu_{2p}\,T_{2p}\left(\frac{\gamma}{\pi}\right)^{2p-1}-{}_nu_{2p+1}\,T_{2p+1}\left(\frac{\gamma}{\pi}\right)^{2p}\right\},$$

where ${}_nu_{2p+1}=2\,{}_nu_{2p}$, ${}_nu_{2p+2}=n^2\,{}_nu_{2p}+2^{2p-2}$, and ${}_nu_4=1$. Further $T_q=\sum_0^\infty\frac{1}{(2i+1)^q}$, and its values are known and tabled, while we ourselves have tabled ${}_nu_q$ from $q=4$ to 31. The series, however, does not converge rapidly and can only be used for γ small.

* For the torsion is proportional to the length.

In the latter case, omitting much troublesome algebra, we find:

$$\tau = \frac{Wa}{EI}\left[\tfrac{1}{15} + \tfrac{13}{15}\eta + (\cdot 084{,}033 - \cdot 168{,}066\eta)\gamma - (\cdot 254{,}632 + \cdot 032{,}403\eta)\gamma^2\right.$$

$$\left. + (\cdot 027{,}637 - \cdot 198{,}882\eta)\gamma^3 - (\cdot 084{,}951 - \cdot 130{,}776\eta)\gamma^4 + \text{etc.}\right] \ldots (43\,a)$$

$$= \frac{Wa}{EI}\left[\tfrac{17}{60} + \cdot 042{,}017\gamma - \cdot 262{,}733\gamma^2 - \cdot 022{,}083\gamma^3 - \cdot 052{,}258\gamma^4 - \text{etc.}\right] \ldots (43\,b),$$

when we substitute the uniconstant value $\frac{1}{4}$ for Poisson's ratio η.

The limiting value as $\gamma \to 0$ is therefore

$$\tau = \frac{Wa}{EI} \times \cdot 28333.$$

2°. $\gamma = 3^\circ$. The approximate formula gives $\tau = \frac{Wa}{EI} \times \cdot 28481$, the accurate value has the coefficient $\cdot 28504$.

3°. $\gamma = 6^\circ$. Using the approximate formula (43 b) we get $\tau = \frac{Wa}{EI} \times \cdot 28482$. The accurate value of the coefficient is $\cdot 28577$.

4°. $\gamma = 12^\circ$. The approximate formula (43 b) gives $\tau = \frac{Wa}{EI} \times \cdot 28031$, the accurate value being $\cdot 28461$. Clearly, except for very rough work, the approximate formula cannot be used beyond about 5° or 6°.

In a considerable number of cases the series were summed and the results are stated below. This number could easily be extended, but we found the labour greater than calculating by modern processes the series to even 80 or more terms. Great care has to be taken that the right limits are found for series and terms which give infinite values. These infinities always cancel, but the difference of the two infinities may provide a finite term. Interesting summations bearing on the theory of numbers constantly occur.

5°. $\gamma = 45^\circ$. This case is analytically of some interest for

$$\sum_{i=0}^{\infty} \frac{64}{(m+2)\,m\gamma\,(m^2-4)} = \frac{8}{\pi}\left(\tfrac{1}{2}\log_e 2 - \tfrac{1}{8}\pi + \cdot 15886{,}74780\right),$$

the ten-figure decimal being provided by Glaisher's sum of $\sum_{i=0}^{\infty} \frac{1}{(4i+3)^2}$: see *Proc. Lond. Math. Soc.*, Vol. VIII, p. 203.

6°. $\gamma = 90^\circ$—a beam with right-angled sector as cross-section.

In this case $m = 2\,(2i+1)$, and the series are all expressible in finite terms. Thus, omitting all the laborious algebra,

$$\sum_{i=0}^{\infty} \frac{64}{(m+2)\,m\gamma\,(m^2-4)} - \tan\gamma + \gamma = \tfrac{2}{3}\pi - \frac{1}{\pi}\,8\log_e 2,$$

$$\sum_{i=0}^{\infty} \frac{1}{(m+2)\,m\gamma\,(m^2-9)} = \frac{1}{12\pi}\,(6\log_e 2 - \pi),$$

$$\sum_{i=0}^{\infty} \frac{1}{(m+2)\,m\gamma\,(m^2-1)} = \frac{1}{180\pi}\,(16 - 3\pi - 22\log_e 2).$$

Substituting these in (41), we obtain

$$\tau = \frac{Wa}{EI} \times \sqrt{2} \times \frac{(112\log_e 2 + 4 - 26\pi) + (16\log_e 2 + 52 - 18\pi)\,\eta}{10\pi^2 - 120\log_e 2}.$$

7°. $\gamma=180°$—a semi-circular solid cylinder with flexing force parallel to the diameter of the cross-section. By similar summations we find:

$$\tau=\frac{Wa}{EI}\times\frac{\pi}{\pi^2-8}\,\frac{2+6\eta}{15}.$$

8°. $\gamma=270°$—a solid circular cylinder with a quadrant cut out of it, the flexing force being perpendicular to the bisector of this quadrant:

$$\tau=\frac{Wa}{EI}\times\frac{\sqrt{2}}{315}-\frac{(4556-2352\log_e 2-546\pi)+(3068-336\log_e 2-378\pi)\,\eta}{2\pi^2-7-8\log_e 2}.$$

9°. $\gamma=360°$—a complete solid circular cylinder with a plane radial cut up to its central axis, the flexing force being perpendicular to this plane cut.

We find that
$$\tau=\frac{Wa}{EI}\times\frac{64\,(8+9\eta)}{25\,(9\pi^2-64)}.$$

Collecting all our results we have the following table:

TABLE I. *Torsion in a bent beam which has for cross-section a non-curtate sector.*

Angles of sector	Value of $\tau\Big/\frac{Wa}{EI}$ for any η	$\eta=\frac{1}{4}$		$\eta=\frac{1}{3}$		$\eta=\frac{1}{2}$	
		$\tau\Big/\frac{Wa}{EI}$	Total torsion / Wa/EI	$\tau\Big/\frac{Wa}{EI}$	Total torsion / Wa/EI	$\tau\Big/\frac{Wa}{EI}$	Total torsion / Wa/EI
0°	·066,667 + ·866,667η	·283,333	− ·158,333	·355,556	− ·188,889	·500,000	− ·250,000
3°	·070,565 + ·857,911η	·285,043	− ·160,043	·356,535	− ·189,868	·499,520	− ·249,520
4°·5476	Interpolated max. $\eta=\frac{1}{3}$	—	—	·356,652	− ·189,985	—	—
6°	·073,458 + ·849,257η	·285,772	− ·160,772	·356,544	− ·189,877	·498,086	− ·248,086
6°·9426	Interpolated max. $\eta=\frac{1}{4}$	·285,816	− ·160,816	—	—	—	—
12°	·076,642 + ·831,860η	·284,607	− ·159,607	·353,929	− ·187,262	·492,572	− ·242,572
18°	·076,772 + ·814,098η	·280,297	− ·155,297	·348,138	− ·181,471	·483,821	− ·233,821
24°	·074,327 + ·795,822η	·273,283	− ·145,283	·339,601	− ·172,934	·472,238	− ·222,238
30°	·069,762 + ·776,990η	·264,010	− ·139,010	·328,759	− ·162,092	·458,257	− ·208,257
45°	·051,839 + ·728,126η	·233,870	− ·108,870	·294,548	− ·127,881	·415,902	− ·165,902
54°	·038,645 + ·698,488η	·213,267	− ·088,267	·271,475	− ·104,808	·387,889	− ·137,889
60°	·029,643 + ·679,077η	·199,412	− ·074,412	·256,002	− ·089,335	·369,181	− ·119,181
75°	·009,098 + ·633,707η	·167,524	− ·042,524	·220,333	− ·053,666	·325,951	− ·075,951
90°	− ·004,459 + ·596,154η	·144,580	− ·019,580	·194,259	− ·027,592	·293,618	− ·043,618
104°·8863	Interpolated min. $\eta=\frac{1}{4}$	·136,646	− ·011,646	—	—	—	—
106°·1906	" " $\eta=\frac{1}{3}$	—	—	·184,139	− ·017,472	—	—
108°·3252	" " $\eta=\frac{1}{2}$	—	—	—	—	·278,781	− ·028,781
120°	·006,222 + ·558,655η	·145,886	− ·020,886	·192,440	− ·025,773	·285,549	− ·035,549
150°	·082,335 + ·584,380η	·228,430	− ·103,430	·277,128	− ·100,461	·374,525	− ·124,525
180°	·224,047 + ·672,141η	·392,082	− ·267,082	·448,094	− ·281,427	·560,117	− ·310,117
210°	·407,389 + ·798,737η	·607,073	− ·482,073	·673,635	− ·506,968	·806,758	− ·556,758
240°	·597,136 + ·929,650η	·829,549	− ·704,549	·907,019	− ·740,352	1·061,961	− ·811,961
270°	·755,377 + 1·028,203η	1·012,428	− ·887,428	1·098,111	− ·831,444	1·269,478	− 1·019,478
300°	·852,715 + 1·066,425η	1·119,321	− ·994,321	1·208,190	− 1·041,423	1·385,928	− 1·135,928
319°·4474	Interpolated max. $\eta=\frac{1}{4}$	1·138,590	− 1·013,590	—	—	—	—
318°·2777	" " $\eta=\frac{1}{3}$	—	—	1·226,331	− 1·059,664	—	—
316°·3202	" " $\eta=\frac{1}{2}$	—	—	—	—	1·402,199	− 1·152,199
330°	·875,056 + 1·031,731η	1·132,989	− 1·007,989	1·218,967	− 1·052,300	1·390,922	− 1·140,922
360°	·824,927 + ·928,043η	1·056,938	− ·931,938	1·134,275	− ·967,608	1·288,948	− 1·038,948

The total torsion = anticlastic torsion minus shearing torsion, and it tends to move the angle edge of prism in the opposite direction to the flexing force. Since the total torsion is always *negative* the angle edge of prism will always move in direction of flexing force.

The curve in Fig. 3 shows how the nature of the torsion changes with γ. It provides a kind of physical measure of the asymmetry of the circular sector, for, as we have seen, the effect is wholly due to the distribution of shearing forces over an asymmetrical section. Remembering the direction of the torsion as given by equation (42) and Fig. 2, we see that in the case of a wedge-shaped sector of small angle, such as the blade of a sword held horizontally*, the torsion will be such that the back twists *up* with a downward flexing force. A first maximum is reached

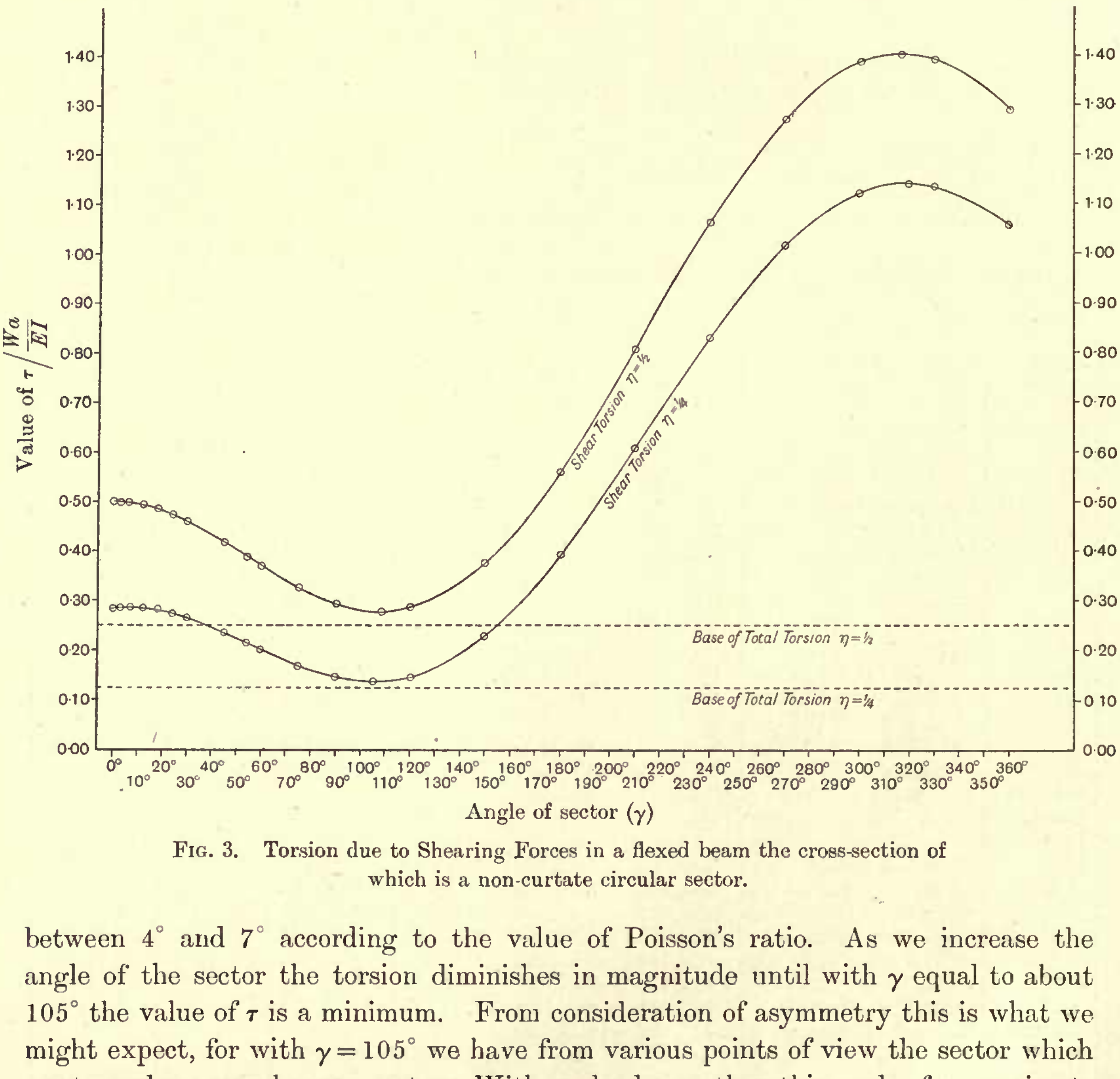


FIG. 3. Torsion due to Shearing Forces in a flexed beam the cross-section of which is a non-curtate circular sector.

between 4° and 7° according to the value of Poisson's ratio. As we increase the angle of the sector the torsion diminishes in magnitude until with γ equal to about 105° the value of τ is a minimum. From consideration of asymmetry this is what we might expect, for with $\gamma = 105°$ we have from various points of view the sector which most nearly approaches symmetry. With angles larger than this angle of approximate symmetry the torsion of the beam increases again. With the increasing departure

* A rough experiment was made with a carving-knife to which a long light pointer was attached to show the torsion when a weight was hung on the end of the blade. The torsion was of course extremely small, but was seen to be in the direction given by the theory.

from symmetry as γ becomes larger, the amount of torsion continues to increase until a second maximum is reached about 316° to 319°, from which value it falls slightly until $\gamma = 2\pi$ which is the physical limit to the application of the formula.

§ 11. *Numerical examples of the non-curtate sector.*

The total torsion in a beam of length l is given by

$$\tau l = \frac{Wal}{EI} \times \text{the constant tabled in Table I}$$

$$= \frac{Wl^3}{3EI} \times \frac{3a}{l^2} \times \text{the tabled constant}$$

$$= \delta \times \frac{3a}{l^2} \times \text{the tabled constant,}$$

if δ be the deflection of the centroid of the end section. This is on the assumption that the length l is so great that we can neglect the linear term $s_0 l$ (see equation (50) *infra*) which comes into the formula for the deflection owing to the shearing forces, an assumption which we shall investigate later in the paper*.

TABLE II. *The deflection and torsion of steel beams of circular sector cross-section with a load W lbs., for varying size of angle.*

Length of beam = 20 inches, breadth = 1 inch, stretch-modulus = 30,000,000 lbs. per square inch.

Angle of sector $=\gamma$	Deflection $\delta = \frac{Wl^3}{3EI}$ inches	Shear torsion $=\tau l$ degrees	Shear torsion per deflection in degrees per inch
1°	$W \times$ 802·53	$W \times$ 97·9356	0°·122
3°	× 29·73	× 3·6416	0°·122
6°	× 3·717	× 0·45646	0°·123
12°	× 0·465	× 0·05686	0°·122
18°	× 0·138	× 0·01662	0°·120
24°	× 0·0586	× 0·00688	0°·117
30°	× 0·0301	× 0·003,415	0°·113
45°	× 0·00908	× 0·000,9123	0°·100
60°	× 0·00393	× 0·000,3368	0°·086
90°	× 0·001,246	× 0·0000,7741	0°·062
120°	× 0·000,5789	× 0·0000,3629	0°·063
150°	× 0·000,3357	× 0·0000,3295	0°·098
180°	× 0·000,2264	× 0·0000,3815	0°·168
210°	× 0·000,1707	× 0·0000,4453	0°·261
240°	× 0·000,1407	× 0·0000,5016	0°·356
270°	× 0·000,1245	× 0·0000,5417	0°·435
300°	× 0·000,1165	× 0·0000,5604	0°·481
330°	× 0·000,1136	× 0·0000,5531	0°·487
360°	× 0·000,1132	× 0·0000,5141	0°·454

NOTE. A positive torsion signifies a torsion in the *opposite* direction to the arrow in Fig. 2, p. 20.

In Table II we have as a particular case the data for a steel beam of length 20 inches and radius 1 inch, the load applied being W. Taking E to be 30,000,000 lbs. per square inch and calculating I from the formula $\frac{a^4}{8}(\gamma - \sin\gamma)$ we get the deflection

* See § 14.

and the torsion of the beam. For another beam of length l' inches, of radius a' inches, and stretch-modulus E', the deflection is got by multiplying the deflections of Table II by $3{,}750 \dfrac{l'^3}{E'a'^4}$ and the total torsion by multiplying the tabled values by $1{,}500{,}000 \dfrac{l'}{E'a'^3}$.

§ 12. *Distribution of vertical shear along the central radius of a cross-section.*

To get some idea of the distribution of shears which gives rise to the torsion, we will consider the distribution of the vertical shear along the central radius of any cross-section, that is the shear $\widehat{zx}$ along the line $x=0$ in any cross-section*. We shall confine ourselves to the case of the complete sector.

From equations (16) and (40),

$$\widehat{zx} = \mu\left\{-\tau y + \frac{\partial\chi}{\partial x} - \frac{W}{EI}\left(\tfrac{1}{2}\eta x^2 + (1-\tfrac{1}{2}\eta)y^2\right) + \beta - \beta''\right\}$$

$$= \mu\left\{-\tau y - \frac{W}{EI}\left(\tfrac{1}{2}\eta x^2 + (1-\tfrac{1}{2}\eta)y^2\right) + \beta - \beta''\right.$$

$$+ \frac{\partial}{\partial x}\left[\Sigma A_m r^m \sin m\theta - (\beta-\beta'') r\sin\theta + \frac{\tau}{2\cos\gamma} r^2 \sin 2\theta\right.$$

$$\left.\left. + \frac{W}{3EI}\left(\tfrac{3}{4} + (\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right) r^3 \sin 3\theta\right]\right\} \quad \text{.........................(44).}$$

Along $x=0$, or $\theta=0$ we have

$$\widehat{zx} = \mu\left\{-\tau y - \frac{W}{EI}(1-\tfrac{1}{2}\eta)y^2 + \Sigma A_m m y^{m-1} + \frac{\tau y}{\cos\gamma} + \frac{W}{EI}\left(\tfrac{3}{4} + (\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right)y^2\right\}$$

$$= \mu\left\{\tau y\left(\frac{1}{\cos\gamma} - 1\right) + \frac{W}{EI}\left(-\tfrac{1}{8} + \tfrac{1}{8}\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right)y^2\right.$$

$$\left. + \Sigma\left(-\frac{8\tau(-1)^i}{(m^2-4)\gamma}\frac{1}{a^{m-2}} + \frac{(-1)^i W}{EI}\frac{\cos\frac{1}{2}\gamma}{\gamma}\left(\frac{7}{2(m^2-1)} - \frac{3}{2(m^2-9)}\right)\frac{1}{a^{m-1}}\right)y^{m-1}\right\},$$

when we put $\eta = \frac{1}{4}$ and insert the expression for A_m,

$$= \mu\frac{Wa^2}{EI}\left\{\left(\tau\Big/\frac{Wa}{EI}\right)\frac{y}{a}\left(\frac{1}{\cos\gamma} - 1\right) + \frac{1}{8}\left(\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma} - 1\right)\frac{y^2}{a^2}\right.$$

$$\left. + \sum_{i=0}^{\infty}\left(-\frac{8\tau\Big/\frac{Wa}{EI}}{m^2-4} + \cos\tfrac{1}{2}\gamma\left(\frac{7}{2(m^2-1)} - \frac{3}{2(m^2-9)}\right)\right)\frac{1}{\gamma}\left(\frac{y}{a}\right)^{m-1}(-1)^i\right\} \text{...(45),}$$

where $m = (2i+1)\dfrac{\pi}{\gamma}$.

* Shears other than in the vertical direction will of course have appreciable and in certain cases may be of great effect on the value of τ, but the labour of calculation would be considerable except in the case we have taken, and that, after all, is the one likely to be most instructive. An important result also flows from this central radius shear: it enables us to determine (see p. 44) the shear at the centroid which provides the linear term of the deflection with Saint-Venant's form of the terminal fixing.

The form of equation (45) allows us immediately to compare the distribution of shear in beams of various angles, but of constant length and radius, when the deflection $\frac{1}{3}\frac{Wl^3}{EI}$ is kept constant. This is the suitable method of comparison, and in what follows we will only consider $\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right)$, that is the expression in the double brackets, it being proportional to $\dfrac{\text{the vertical shear on central axis}}{\text{deflection}}$.

It is useless to attempt to see the nature of the distribution of shear from the equation itself, and we have immediately to resort to numerical calculation.

In Table III below are given the values of $\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right)$ for every tenth of the radius of the sector and for sectors whose angles are 5°, 12°, 30°, 45°, 60°, 72°, 90°, 120°, 150°, 180°, 270°, 360°.

For all these values of γ the series involved in equation (45) may each be summed in finite terms. For let us consider the series

$$\sum_{i=0}^{\infty}\frac{1}{m^2-p^2}\left(\frac{y}{a}\right)^{m-1}(-1)^i,$$

a series which for $p=1, 2, 3$ gives the three series concerned.

$$\begin{aligned}
\Sigma\frac{1}{m^2-p^2}\left(\frac{y}{a}\right)^{m-1}(-1)^i &= \frac{1}{2p}\left[\Sigma\frac{1}{m-p}\left(\frac{y}{a}\right)^{m-1}(-1)^i - \Sigma\frac{1}{m+p}\left(\frac{y}{a}\right)^{m-1}(-1)^i\right]\\
&= \frac{1}{2p}\left[\left(\frac{y}{a}\right)^{p-1}\Sigma\frac{1}{m-p}\left(\frac{y}{a}\right)^{m-p}(-1)^i - \left(\frac{y}{a}\right)^{-p-1}\Sigma\frac{1}{m+p}\left(\frac{y}{a}\right)^{m+p}(-1)^i\right]\\
&= \frac{1}{2p}\left[\left(\frac{y}{a}\right)^{p-1}\Sigma\int_0^{\frac{y}{a}}\left(\frac{y}{a}\right)^{m-p-1}(-1)^i\,d\left(\frac{y}{a}\right) - \left(\frac{y}{a}\right)^{-p-1}\Sigma\int_0^{\frac{y}{a}}\left(\frac{y}{a}\right)^{m+p-1}(-1)^i\,d\left(\frac{y}{a}\right)\right]\\
&= \frac{1}{2p}\left[\left(\frac{y}{a}\right)^{p-1}\int_0^{\frac{y}{a}}\sum_{i=0}^{\infty}\left(\frac{y}{a}\right)^{(2i+1)\frac{\pi}{\gamma}-p-1}(-1)^i\,d\left(\frac{y}{a}\right)\right.\\
&\qquad\left. - \left(\frac{y}{a}\right)^{-p-1}\int_0^{\frac{y}{a}}\sum_{i=0}^{\infty}\left(\frac{y}{a}\right)^{(2i+1)\frac{\pi}{\gamma}+p-1}(-1)^i\,d\left(\frac{y}{a}\right)\right],
\end{aligned}$$

the interchange of the integral and summation signs being clearly justified,

$$=\frac{1}{2p}\left[\left(\frac{y}{a}\right)^{p-1}\int_0^{\frac{y}{a}}\frac{\left(\frac{y}{a}\right)^{\frac{\pi}{\gamma}-p-1}}{1+\left(\frac{y}{a}\right)^{\frac{2\pi}{\gamma}}}\,d\left(\frac{y}{a}\right) - \left(\frac{y}{a}\right)^{-p-1}\int_0^{\frac{y}{a}}\frac{\left(\frac{y}{a}\right)^{\frac{\pi}{\gamma}+p-1}}{1+\left(\frac{y}{a}\right)^{\frac{2\pi}{\gamma}}}\,d\left(\frac{y}{a}\right)\right]\text{......(46)},$$

and both of these integrands being rational functions of $\sqrt{\frac{y}{a}}$, *when* $\frac{2\pi}{\gamma}$ *is an integer* (as in all the cases considered), we must be able to express the series in finite terms by evaluating these integrals. In general they are incomplete B-functions.

For the sectors whose angles are 5°, 12°, 30°, 45°, 60°, 72° and 120°, it is easier to use the series as they stand than to find the sums in finite terms and calculate from them. For these cases the equation (45) becomes:

for 5°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 001091\frac{y}{a} + \cdot 000959\left(\frac{y}{a}\right)^2 - \cdot 002668\left(\frac{y}{a}\right)^{35} + \cdot 000283\left(\frac{y}{a}\right)^{107} + \ldots,$$

for 12°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 006358\frac{y}{a} + \cdot 005713\left(\frac{y}{a}\right)^2 - \cdot 007971\left(\frac{y}{a}\right)^{14}$$
$$+\cdot 000701\left(\frac{y}{a}\right)^{44} - \cdot 000247\left(\frac{y}{a}\right)^{74} + \cdot 000128\left(\frac{y}{a}\right)^{104} + \ldots,$$

for 30°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 040842\frac{y}{a} + \cdot 045753\left(\frac{y}{a}\right)^2 - \cdot 044065\left(\frac{y}{a}\right)^5$$
$$+\cdot 001400\left(\frac{y}{a}\right)^{17} - \cdot 000426\left(\frac{y}{a}\right)^{29} - \ldots,$$

for 45°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 096872\frac{y}{a} + \cdot 176777\left(\frac{y}{a}\right)^2 - \cdot 176109\left(\frac{y}{a}\right)^3$$
$$+\cdot 001295\left(\frac{y}{a}\right)^{11} - \cdot 000210\left(\frac{y}{a}\right)^{19} + \cdot 000073\left(\frac{y}{a}\right)^{27} + \ldots,$$

for 60°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 199412\frac{y}{a} - \cdot 033411\left(\frac{y}{a}\right)^2 - \cdot 4760556\left(\frac{y}{a}\right)^2\log_{10}\frac{y}{a} + \cdot 000832\left(\frac{y}{a}\right)^8$$
$$+\cdot 000286\left(\frac{y}{a}\right)^{14} - \cdot 000221\left(\frac{y}{a}\right)^{20} + \cdot 000152\left(\frac{y}{a}\right)^{26} - \cdot 000108\left(\frac{y}{a}\right)^{32} + \ldots,$$

for 72°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = +\cdot 38699\frac{y}{a} - \cdot 452254\left(\frac{y}{a}\right)^2$$
$$+\sqrt{\frac{y}{a}}\left(\cdot 289781\frac{y}{a} + \cdot 000780\left(\frac{y}{a}\right)^6 + \cdot 000706\left(\frac{y}{a}\right)^{11} - \cdot 000481\left(\frac{y}{a}\right)^{16} + \ldots\right),$$

for 120°, $$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = -\cdot 437658\frac{y}{a} - \cdot 187500\left(\frac{y}{a}\right)^2$$
$$+\sqrt{\frac{y}{a}}\left(1\cdot 071134 + \cdot 022715\left(\frac{y}{a}\right)^3 - \cdot 003120\left(\frac{y}{a}\right)^6 + \cdot 001121\left(\frac{y}{a}\right)^9\right.$$
$$-\cdot 000579\left(\frac{y}{a}\right)^{12} + \cdot 000355\left(\frac{y}{a}\right)^{15} - \cdot 000241\left(\frac{y}{a}\right)^{18} + \cdot 000209\left(\frac{y}{a}\right)^{21}$$
$$-\cdot 000134\left(\frac{y}{a}\right)^{24} + \cdot 000105\left(\frac{y}{a}\right)^{27} - \cdot 000084\left(\frac{y}{a}\right)^{30} + \cdot 000070\left(\frac{y}{a}\right)^{33}$$
$$\left.-\cdot 000059\left(\frac{y}{a}\right)^{36} + \cdot 000050\left(\frac{y}{a}\right)^{39} - \cdot 000043\left(\frac{y}{a}\right)^{41} + \ldots\right).$$

For the sectors whose angles are 90°, 180°, 360° the series are too slowly convergent for calculation, but the sums of the series are without much difficulty obtainable by the method used to obtain equation (46).

For $\gamma = 90^\circ$:

$$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right)=\left(\tau\Big/\frac{Wa}{EI}\right)\left[-\frac{y}{a}+\left(\frac{y}{a}+\left(\frac{y}{a}\right)^{-3}\right)\log\left(1+\left(\frac{y}{a}\right)^4\right)\frac{1}{\pi}-\frac{4}{\pi}\frac{y}{a}\log_e\frac{y}{a}\right]-\frac{1}{4}\left(\frac{y}{a}\right)^2$$

$$+\frac{\cos\frac{1}{2}\gamma}{\gamma}\left[\left(\frac{7}{4}-\frac{1}{4}\left(\frac{y}{a}\right)^{-4}\right)\left(\frac{1}{4\sqrt{2}}\log_e\frac{\left(\frac{y}{a}\right)^2+\frac{y}{a}\sqrt{2}+1}{\left(\frac{y}{a}\right)^2-\frac{y}{a}\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\frac{y}{a}\sqrt{2}}{1-\left(\frac{y}{a}\right)^2}\right)\right.$$

$$+\left(\frac{7}{4}\left(\frac{y}{a}\right)^{-2}-\frac{1}{4}\left(\frac{y}{a}\right)^2\right)\left(\frac{1}{4\sqrt{2}}\log_e\frac{\left(\frac{y}{a}\right)^2+\left(\frac{y}{a}\right)\sqrt{2}+1}{\left(\frac{y}{a}\right)^2-\left(\frac{y}{a}\right)\sqrt{2}+1}-\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\frac{y}{a}\sqrt{2}}{1-\left(\frac{y}{a}\right)^2}\right)$$

$$\left.+\frac{1}{4}\left(\frac{y}{a}+\left(\frac{y}{a}\right)^{-3}\right)\right] \qquad \text{.......(47).}$$

Instead of this complicated expression the series formula of equation (45) may be used directly for low values of $\frac{y}{a}$. Actually, in the calculations for the table given on p. 31, this method was used as a check on the accuracy of the work.

For $\gamma = 180^\circ$, the case of a semi-cylinder flexed in the plane parallel to its plane side:

$$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right)=\frac{7}{8}-\frac{1}{8}\left(\frac{y}{a}\right)^2+\left(\tau\Big/\frac{Wa}{EI}\right)\left[-2\frac{y}{a}+\frac{2}{\pi}\left(1+\frac{y}{a}\tan^{-1}\frac{y}{a}+\left(\frac{y}{a}\right)^{-3}\left(\frac{y}{a}-\tan^{-1}\frac{y}{a}\right)\right)\right] \qquad \text{.......(48).}$$

For $\gamma = 270^\circ$:

$$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right)=-\left(\tau\Big/\frac{Wa}{EI}\right)\left[\frac{y}{a}+\frac{1}{\pi}\left\{-\left(\frac{y}{a}\right)^{\frac{1}{3}}\left(1+\left(\frac{y}{a}\right)^{-2}\right)\right.\right.$$
$$\left.\left.-\frac{4}{3}\frac{y}{a}\log_e\frac{y}{a}+\left(\frac{y}{a}+\left(\frac{y}{a}\right)^{-3}\right)\log_e\left(1+\left(\frac{y}{a}\right)^{\frac{4}{3}}\right)\right\}\right]$$
$$-\frac{1}{4}\left(\frac{y}{a}\right)^2+\frac{1}{8\pi}\left(\left(\frac{y}{a}\right)^{-4}-7\right)\left\{\frac{1}{2}\log_e P-\cot^{-1}Q'\right\}$$
$$+\frac{1}{8\pi}\left(7\left(\frac{y}{a}\right)^{-2}+\left(\frac{y}{a}\right)^2\right)\left\{\frac{1}{2}\log_e P+\cot^{-1}Q'\right\}$$
$$+\frac{1}{2\pi\sqrt{2}}\left\{\left(\frac{y}{a}\right)^{-\frac{1}{3}}\frac{48-50\left(\frac{y}{a}\right)^{-\frac{4}{3}}}{7}+\frac{1}{3}\left(\frac{y}{a}+\left(\frac{y}{a}\right)^{-3}\right)\right\},$$

where $$P=\left\{\frac{1}{\sqrt{2}}\left(\left(\frac{y}{a}\right)^{-\frac{1}{3}}+\left(\frac{y}{a}\right)^{\frac{1}{3}}\right)+1\right\}\Big/\left\{\frac{1}{\sqrt{2}}\left(\left(\frac{y}{a}\right)^{-\frac{1}{3}}+\left(\frac{y}{a}\right)^{\frac{1}{3}}\right)-1\right\}$$

and $$Q'=\left(\left(\frac{y}{a}\right)^{-\frac{1}{3}}-\left(\frac{y}{a}\right)^{\frac{1}{3}}\right)\frac{1}{\sqrt{2}} \qquad \text{.......(49).}$$

For $\gamma = 360^\circ$, the case of a complete cylinder with a radial cut perpendicular to its plane of flexure :

$$\widehat{zx}\Big/\left(\mu\frac{Wa^2}{EI}\right) = -2\left(\tau\Big/\frac{Wa}{EI}\right)\frac{1}{\pi}\left[\sqrt{\frac{y}{a}}\left(1-\frac{1}{3}\left(\frac{y}{a}\right)^{-1}-\frac{1}{3}\left(\frac{y}{a}\right)^{-2}+\left(\frac{y}{a}\right)^{-3}\right)\right.$$
$$\left.+\left(\frac{y}{a}-\frac{y}{a}^{-3}\right)\tan^{-1}\sqrt{\frac{y}{a}}\right]$$
$$-\frac{1}{4\pi}\left[\sqrt{\frac{y}{a}}\left(-\frac{1}{3}+\frac{y}{a}-\frac{34}{5}\left(\frac{y}{a}\right)^{-1}-\frac{34}{5}\left(\frac{y}{a}\right)^{-2}-\frac{1}{3}\left(\frac{y}{a}\right)^{-3}+\left(\frac{y}{a}\right)^{-4}\right)\right.$$
$$\left.+\left(+\left(\frac{y}{a}\right)^2-7+7\left(\frac{y}{a}\right)^{-2}-\left(\frac{y}{a}\right)^{-4}\right)\tan^{-1}\sqrt{\frac{y}{a}}\right]\text{.........(50).}$$

In the cases of $\gamma = 60^\circ$, $\gamma = 90^\circ$, $\gamma = 180^\circ$, m takes the values 3, 2, 1 respectively and consequently there arise infinite terms in the series of equation (45). There are however corresponding trigonometrical infinities of the same order and the limits of the whole expression are, as they must be, finite. For $\gamma > 180^\circ$ the shear becomes infinite at $\frac{y}{a} = 0$. This is the case of the re-entrant angle referred to later on in this section.

The accompanying Table III gives the collected numerical results and in Fig. 4 are diagrams each representing the distribution of shear along the central radius for different sectorial angles. It would be impossible to show adequately in one diagram the comparative distributions of stress for the different sizes of the sectorial angle. It is to be remembered in considering the table and the diagrams that, in accordance with the axes of reference which we have taken, a shear of the material on the fixed side on material on the loaded side of the cross-section is positive when it acts opposite to the direction of load.

Our diagrams provide the excess of actual shear over mean shear in terms of mean shear. They indicate the point at which the shear is exactly equal to the mean shear and the value of the shear at the centroid. When $\gamma = 0$ we have, if A stand for area of section and W/A for mean shear,

$$\widehat{zx}/(W/A) = 1{\cdot}36\,\frac{y}{a} + 1{\cdot}2\left(\frac{y}{a}\right)^2,$$

or the limiting case is a parabola. The ratio runs from 0 to 2·56, or the excess on mean shear from -1 to 1·56. The curve is given in Fig. 4 (i) of this series. Almost at the half radius (·508) the shear is equal to the mean shear, but on the right of this point, i.e. towards the thick end, the excesses on the mean shear are considerably greater than the defects on the left, and we get light on why the thick end rotates upwards. We may look upon this curve as the parabola plus the vertical line through the point 1·0. This explains how the apex curve comes away as in Fig. 4 (ii) of the series, when $\gamma = 5^\circ$, and the vertical line bends round so that we actually get *negative* central shears at the butt end in sector prisms of very small angle. A horizontal line through $-1{\cdot}0$ of vertical scale indicates in all these diagrams the extent of the

TABLE III. *Values of the shear along the central radius of symmetry and parallel to the flexing load in a series of sections.*

In Section A of the Table the values of $\widehat{zx}\Big/\left(\mu\dfrac{Wa^2}{EI}\right)$ are recorded; in Section B the values of $\widehat{zx}$/(mean shear) − 1.

Angle of Sector

A	y/a	5°	12°	30°	45°	60°	72°	90°	120°	180°	270°	360°	y/a
$\widehat{zx}\Big/\left(\mu\dfrac{Wa^2}{EI}\right)$	0·0	·00000	·00000	·00000	·00000	·00000	·00000	·00000	·00000	1·27369	∞	∞	0·0
	0·1	·00012	·00069	·00454	·01128	·02437	·04341	·09602‡	·29309	1·18088	1·62243	3·28157	0·1
	0·2	·00026	·00150	·00998	·02504	·05186	·08537	·16159	·38408	1·09029	1·19671	2·20488	0·2
	0·3	·00041	·00242	·01626	·04022	·07922	·12322	·21250	·43885	1·00171	·94698	1·67942	0·3
	0·4	·00059	·00346	·02321	·05576	·10473	·15603	·25214	·47329	·91485	·75778	1·32753	0·4
	0·5	·00079	·00461	·03048	·07062	·12718	·18325	·28223	·49368	·82935	·59358	1·05453	0·5
	0·6	·00100	·00587	·03755	·08373	·14565	·20452	·30367	·50330	·74518	·45038	·82407	0·6
	0·7	·00123	·00720	·04361	·09405	·15940	·21959	·31717	·50418	·60085	·31429	·61890	0·7
	0·8	·00149	·00839	·04755	·10058	·16782	·22829	·32327	·49770	·57712	·15781	·42969	0·8
	0·9	·00169	·00853	·04801	·10237	·17045	·23054	·32244	·48484	·49310	·05215	·25844	0·9
	1·0	− ·00034*	·00443	·04351	·09870	·16694	·22623	·31512	·46634	·40864	− ·07666	·07899	1·0
B	y/a	**5°**	**12°**	**30°**	**45°**	**60°**	**72°**	**90°**	**120°**	**180°**	**270°**	**360°**	y/a
$\widehat{zx}$/(mean shear) − 1	0·0	− 1·00000	− 1·00000	− 1·00000	− 1·00000	− 1·00000	− 1·00000	− 1·00000	− 1·00000	·93250	∞	∞	0·0
	0·1	− ·85044	− ·84800	− ·83888	− ·81897	− ·77464	− ·71437	− ·57722§	− ·20044	·80820	1·14146	4·25051	0·1
	0·2	− ·67641	− ·67095	− ·64556	− ·59814	− ·52043	− ·43830	− ·28851	·04777	·68622	·57955	2·52781	0·2
	0·3	− ·47842	− ·46886	− ·42264	− ·35449	− ·26736	− ·18923	− ·06434	·19719	·56626	·24992	1·68708	0·3
	0·4	− ·25622	− ·24170	− ·17617	− ·10497	− ·03143	·02662	·11019	·29116	·44794	·00020	1·12405	0·4
	0·5	− ·00980	·01045	·08213	·13346	·17622	·20571	·24269	·34677	·33077	− ·21652	·68724	0·5
	0·6	·26070	·28646	·33304	·34390	·34703	·34565	·33707	·37301	·21423	− ·40554	·31851	0·6
	0·7	·55554	·58702	·54802	·50959	·47416	·44480	·39651	·37543	·09779	− ·58517	− ·00976	0·7
	0·8	·88682	·84072	·68794	·61431	·55206	·50208	·42338	·35775	− ·01900	− ·79171	− ·31250	0·8
	0·9	1·13324	·87162	·70442	·64316	·57640	·51688	·41975	·32267	− ·13684	− ·93112	− ·58650	0·9
	1·0	− 1·42259†	− ·02327	·54444	·58417	·54391	·48852	·38752	·27220	− ·25592	− 1·10119	− ·87361	1·0

* 0·95 : ·00141; 0·98 : ·00069; 0·99 : ·00021.

† 0·95 : ·78089; 0·98 : − ·12945; 0·99 : − ·73425.

‡ 0·05 : ·05307.

§ 0·05 : − ·76643.

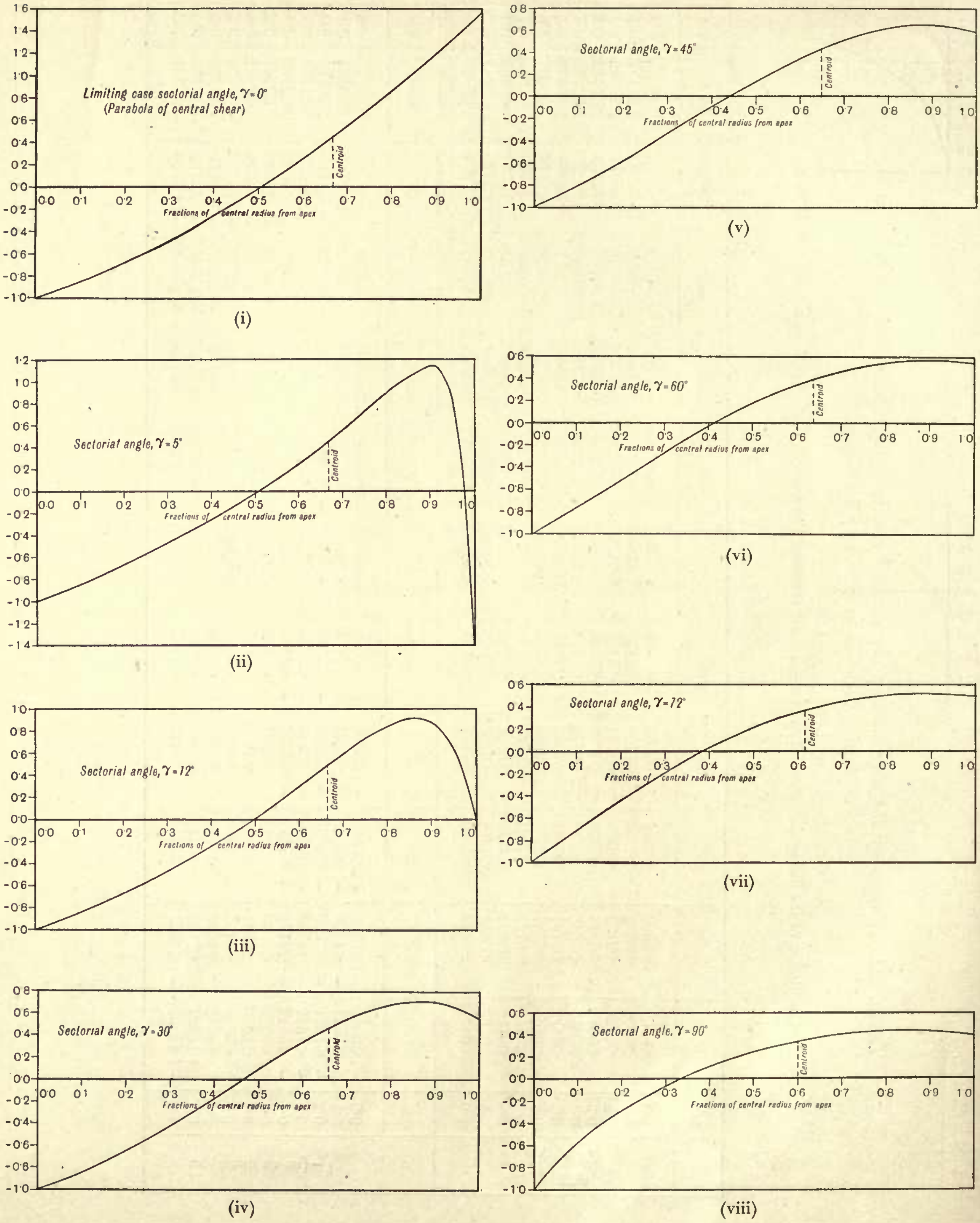
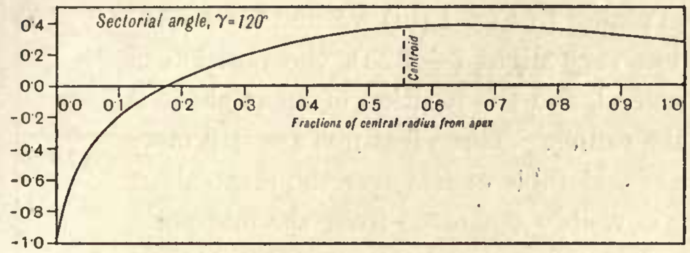
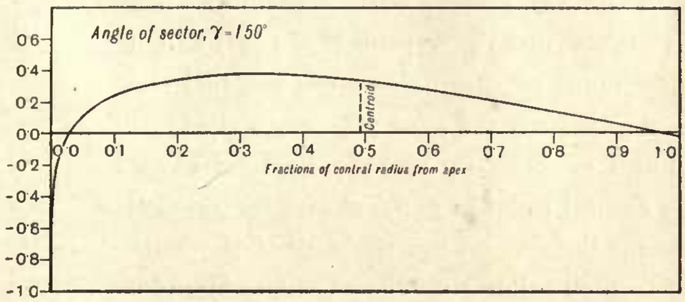


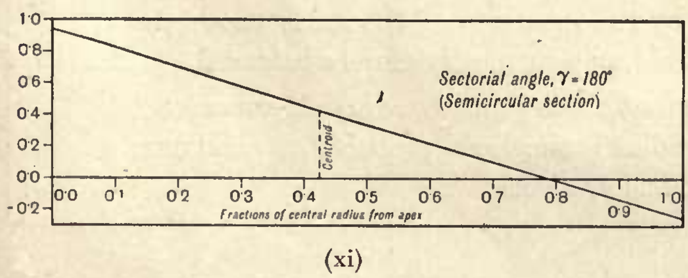
FIG. 4. Diagrams showing the Distribution of Vertical Shear along the central radius of



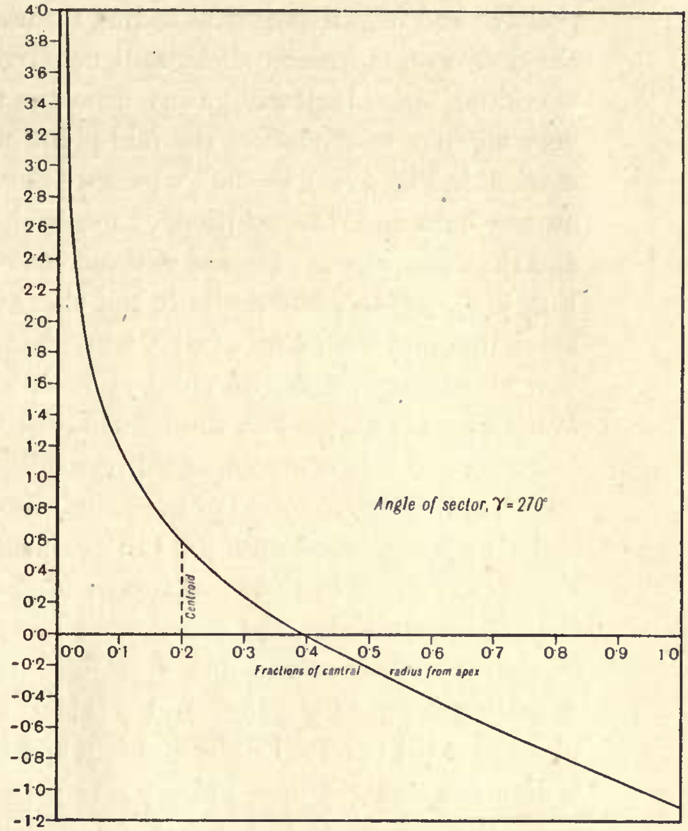
(ix)



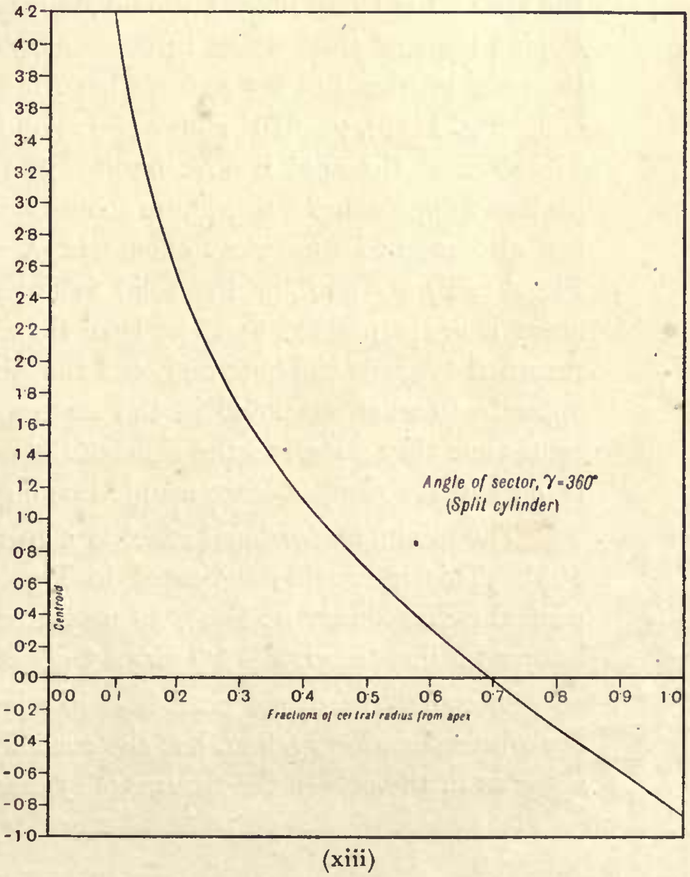
(x)



(xi)



(xii)



(xiii)

a flexed beam having a complete circular sector cross-section, γ being the sectorial angle.

positive and negative central radius shears. Passing to Fig. 4 (iii) we find for $\gamma = 12^\circ$ the defect on the mean shear still negative but very slight ($-\cdot 023$), the position of maximum shear is passing away from the butt end, and the position of mean shear is very slightly to the left of the mid-point of the radius. These features are still more marked in Fig. 4 (iv), $\gamma = 30^\circ$, where we have a considerable excess over the mean shear at the butt end, the position of mean shear is visibly separated from the mid-point and the shear at the butt end still increasing. The process is continued still further in Fig. 4 (v), $\gamma = 45^\circ$, but the butt end shear in terms of mean shear has now reached a maximum and with Fig. 4 (vi), $\gamma = 60^\circ$, has begun to fall. The shear at the centroid is now also falling; and the point of mean shear is moving rapidly towards the apex. With Fig. 4 (vii), $\gamma = 72^\circ$, the tendency of the curve to flatten out horizontally is conspicuous and this is accompanied by a fall in the butt end shear and the centroid shear with a still further motion of the point of mean shear towards the apex. In Fig. 4 (viii) and (ix) for $\gamma = 90^\circ$ and $\gamma = 120^\circ$ respectively these features are emphasised. In Fig. 4 (x) for $\gamma = 150^\circ$ we are approaching a climax. The butt end shear is now less than the mean shear, the position of mean almost up to the apex and the flattening of the top of the curve now stretches across from 0·2 to 0·5 almost of the radius. A section between $\gamma = 150^\circ$ and $\gamma = 180^\circ$ would show the curve with a pointed apex like Fig. 4 (ii) reversed right and left, and this enables us to interpret Fig. 4 (xi) which is almost a straight line. For $\gamma = 180^\circ$ the first point of mean shear has reached the apex, the zero shear has ceased and we have a finite excess over the mean shear at the apex; the curve indeed should be considered as built-up of the nearly straight sloping line and the vertical through 0·0. The second point of mean shear has reached the neighbourhood of 0·8 and we have increasing defects on mean shear at the butt end. Fig. 4 (xii), $\gamma = 270^\circ$, shows the revolution which follows on the re-entrant angle. The shear at the apex is now infinite; the shear at the butt end is now negative and probably approaches a maximum somewhere about 300°. The position of mean shear has also reached somewhat about its maximum in motion towards the right. In Fig. 4 (xiii), $\gamma = 360^\circ$, or the solid cylinder with a cut to the centre, the butt end shear is again positive and less than the mean shear, the position of mean shear has returned towards the butt end, and the big shears are all towards the left, the exact opposite to what we noted in the sectors of small angle first discussed, but at the same time they are all to the right of the centroid, so that their total turning moment is big and the result a large couple turning the butt end upwards.

The minimum torsional effect is shown in Table I to occur in a sector of about 105°. This is roughly indicated in Fig. 4 (viii) and (ix) where the couple of the central radius shears at least, as measured by the moments of the areas above the horizontal line through $-1\cdot 0$ about the centroid, appear mostly closely balanced.

Of course the reader must bear in mind that these central radius shears are not the whole shearing system, but the central radius is the locus probably of maximum shear as in the case of the flexure of symmetrical systems.

§ 13. *The torsion of the curtate sector for particular cases.*

The formulae for the curtate sector are very much more complicated than the formulae for the complete sector and require a large amount of additional arithmetic. This is however much lessened if the case of the complete sector for the angle in question has been worked out, for the calculation then consists of multiplying all the quantities concerned by appropriate factors. These factors, also, converge to unity with great rapidity if the ratio of the inner radius a_0 to the outer radius a is not too large and if the value of γ is fairly small so that m is large. This is the case, for instance, when $\gamma = 12°$, which is given below.

For the limiting case of $a_0 \to a$, we can expand all the factors of the form $a^p - a_0^p$ in powers of $\epsilon = \dfrac{a - a_0}{a_0}$, and we obtain for limiting value

$$\tau = \frac{Wa}{EI}(84{\cdot}56 + 86{\cdot}70\eta) = \frac{Wa}{EI} \times 106{\cdot}24, \text{ if } \eta = \tfrac{1}{4}.$$

1°. *A thin wedge-shaped curtate sector.*

The angle chosen was 12°, and the results of the calculations are given below in Table IV and graphically in Figs. 5 and 6. The column headed $\psi_1 = \tau \Big/ \dfrac{Wa}{EI}$ gives the constants as directly calculated, and the following column headed $\psi_2 = \tau \Big/ \dfrac{W(a - a_0)}{EI}$ is deduced from the previous one. The first expression gives the ratio of total torsion (τl) to deflection per unit length $\left(\dfrac{\delta}{l} = \dfrac{1}{3}\dfrac{Wl^2}{EI}\right) = \dfrac{h}{l} \times \dfrac{3}{2 \sin \frac{1}{2}\gamma}\psi_1$, where h is the maximum height of section and ψ_1 the number in the second column of Table IV. It shows that for constant height of section and length of beam the torsion at first decreases slowly

TABLE IV. *The torsion per unit length in a sector of central angle* 12° *and external radius a for varying degrees of curtateness.*

Internal radius / External radius $= \dfrac{a_0}{a}$	Torsion per unit length	
	$\psi_1 = \tau \Big/ \dfrac{Wa}{EI}$	$\psi_2 = \tau \Big/ \dfrac{W(a-a_0)}{EI}$
0·0	·2846	·2846
·3	·2486	·3729
·6	·2205	·6614
·75	·2219	·8875
·8	·2248	1·1238
·9	·2386	2·3861
·95	·2686	5·3722
·98	·4528	22·6390
1·00	106·24	∞

NOTE. A positive torsion signifies a torsion in the *opposite* direction to the arrow in Fig. 2, p. 20.

and then, after $a_0 = \cdot 9a$, increases rapidly with the squatness. The second expression indicates that the same ratio may be represented by $\frac{a - a_0}{l} \times 3\psi_2$, where ψ_2 is the number in the third column of Table IV, and the values of ψ_2 show that for constant breadth of section and length of beam the ratio of total torsion to deflection per unit length increases at first slowly and then very rapidly with the squatness.

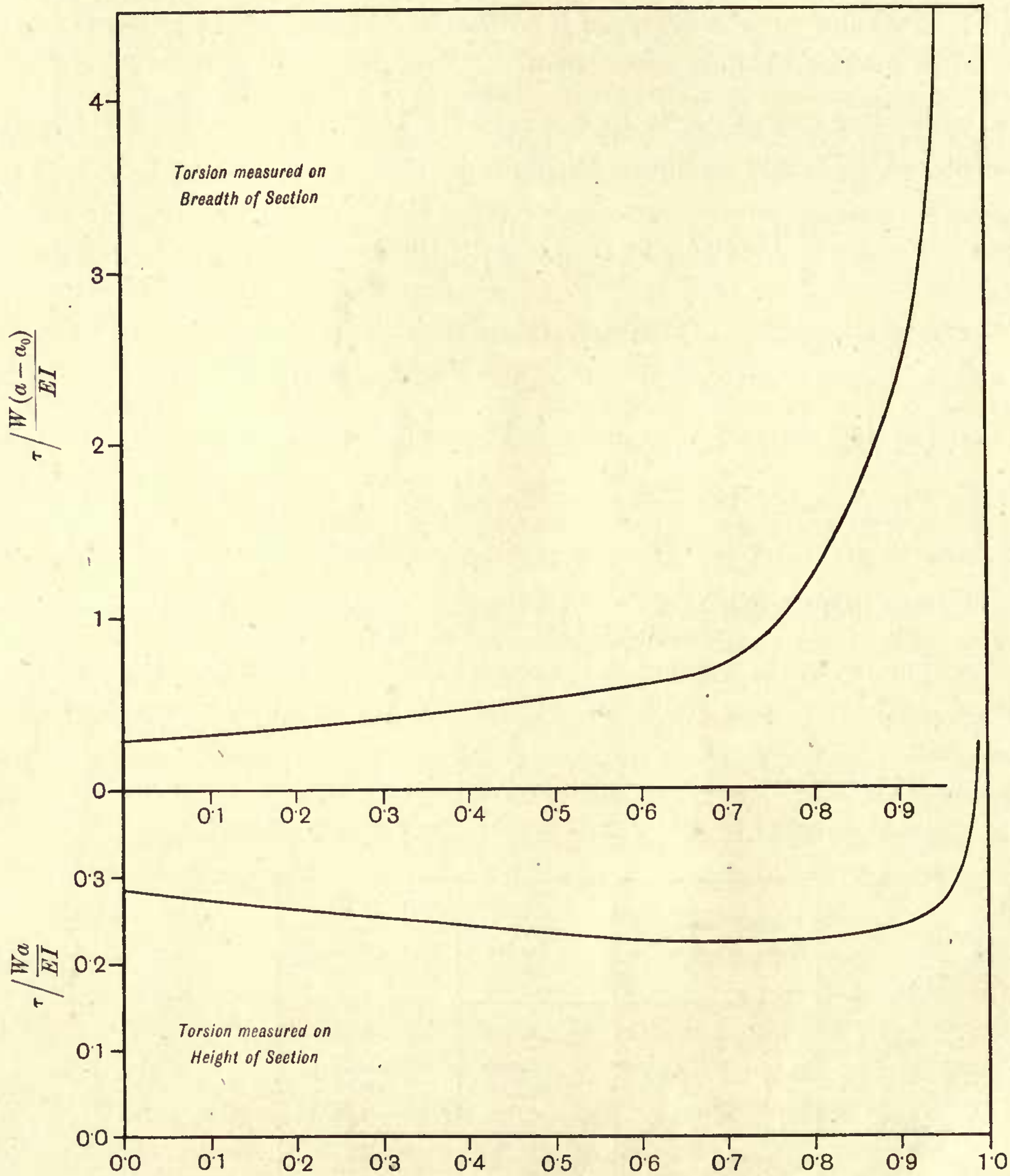


FIG. 5.

FIG. 6.

FIG. 5. The torsion of a flexed beam which has as cross-section a thin (angle = 12°) wedge-shaped sector with varying curtateness, the torsion being measured on the *external* radius.

FIG. 6. Torsion of the same beam as in Fig. 5, the torsion being measured on the central breadth, not the external radius.

Numerical example.

As in the case of the non-curtate sector, we shall now consider the special example of a beam of steel 20 inches long having as cross-section a curtate sector of angle 12° and outer radius 1 inch (the outer arc is therefore in all cases ·209... inches). The values of the deflection and torsion when a load W is applied are given in Table V and the torsion per deflection for varying curtateness is shown in Fig. 6, the scale in degrees per inch being obtained from the scale of $\tau \Big/ \frac{Wa}{EI}$ there given by multiplying by the factor ·430.

TABLE V. *The deflection and torsion of a steel beam of curtate circular sector cross-section.*

Length of beam = 20 inches.
Outer arc = 0·209 inches.
Angle of sector = 12°.
Stretch-modulus = 30,000,000 lbs. per square inch.

$\frac{\text{Internal radius}}{\text{External radius}} = \frac{a_0}{a}$	Deflection $= \frac{1}{3}\frac{Wl^3}{EI}$ inches	Total torsion $=\tau l$ degrees	Torsion (in degrees) per inch of deflection
0	$W \times$ ·465	$W \times$ ·0569°	0°·1223
·3	$\times$ ·471	$\times$ ·0503°	0°·1068
·6	$\times$ ·580	$\times$ ·0549°	0°·0948
·75	$\times$ ·681	$\times$ ·0649°	0°·0954
·8	$\times$ ·788	$\times$ ·0762°	0°·0966
·9	$\times$ 1·353	$\times$ ·1389°	0°·1027
·95	$\times$ 2·509	$\times$ ·2896°	0°·1154
1·0	$\times\ \infty$	$\times\ \infty$	45°·68

NOTE. A positive torsion signifies a torsion in the *opposite* direction to the arrow in Fig. 2, p. 20.

2°. *A cylindrical tube with a longitudinal cut.*

If we take $\gamma = 2\pi$ we get a sector which is a full circular ring except that it has a cut along one radius. The beam is then really a cylindrical tube with a cut along one side. This case, especially if the tube be thin (e.g. as below $a_0/a = ·9$), is of peculiar interest as it represents the extreme of asymmetry in a circular sector. The method of approximation for $a_0 \to a$ mentioned at the beginning of this section having failed, we have to use direct calculation and, both γ and the ratio $\frac{a_0}{a}$ being large, the calculation of the series is extremely heavy. The factor of τ in equation (39) turns out to be the difference of two nearly equal numbers and to get three figure accuracy in the final result it was found necessary to calculate a large number of terms of the series to 10 decimal places.

If we take $a_0 = ·9a$ we get

$$\tau = \frac{Wa}{EI} \times 520·6\,(1+\eta) = \frac{Wa}{EI}\, 650·75, \text{ for } \eta = \tfrac{1}{4},$$

as compared with $$\tau = \frac{Wa}{EI} \times 1·057$$

for a non-curtate sector of the same angle. This is an extraordinarily large value for τ, but it is well known that a hollow rod loses much of its power to resist torsion when it has a cut down one side and it might therefore be anticipated that the couple due to the great asymmetry of the flexural shears would bring about a correspondingly large torsion.

We can illustrate this result by taking the value of τ for $\epsilon = (a - a_0)/a_1$ very small. The deduction of this value is of much interest as it involves the summation of a number of noteworthy series which, however, cannot be considered here; ultimately we find:

$$\tau = \frac{Wa}{EI}\frac{4\cdot 7124\,(1+\eta)}{\epsilon^2} = \frac{Wa}{EI} = \frac{Wa}{EI}\frac{5\cdot 8905}{\epsilon^2},$$

indicating the high value of τ as the thickness diminishes indefinitely*.

Experimental Verification.

With such a large value of τ it is easily possible to see the twisting of such a beam under flexure, and some rough preliminary experiments were made with a piece of piping used as a casing to electric wiring. This is manufactured with the required slit. One end was fixed in the chuck of a turning lathe, a solid boss having been put inside the tube at that end to prevent buckling. The load was applied by a stirrup arrangement fastened to the other end. The deflection was measured by taking the vertical heights of the bottom of tube from the lathe-bed or from the slide-rest. The torsion of the load-end relative to the fixed end was measured with the aid of a light pointer fastened rigidly at the load-end moving over a plate fastened rigidly to the fixed end. The effective radius of the pointer was 10 inches and the torsion was measured on a 12″ length of the 15″ pipe so that with the largest observed torsion of ·159 radians or 9° the pointer moved through 1·59 inches, and the real torsion was $\frac{15}{12}$. 159 = ·199 radians.

The dimensions of the tube were: Outer radius = 0·50 inches, inner radius = 0·45 inches, length (excluding the part used in fixing) = 15 inches.

TABLE VI. *Experiments with a piece of split piping. Deflection and torsion with various loads.*

Load in lbs. $= W$	Experimental			Theoretical ($E = 27{,}000{,}000$)		
	Average deflection at end in inches $= \delta$	Average torsion $= \tau l$		Deflection in inches (linear and cubic terms)	Torsion in radians	
		In radians	In degrees		$a_0 = \cdot 45$	$a_0 = \cdot 425$
0	·000	·000	0° 0′	·000	·000	·000
3·7	·048	·015	0° 52′	·037	·041	·017
13·7	·131	·059	3° 23′	·138	·151	·063
23·7	·215	·105	6° 1′	·239	·262	·108
33·7	·305	·152	8° 43′	·339	·372	·153
43·7	·387	·199	11° 24′	·440	·488	·199

* If ϵ be put 0·1 we have $\tau = \frac{Wa}{EI}\,589\cdot 05$, which is of the same order as the true value above, but indicates that $\frac{1}{10}$ is not small enough for us to neglect the next term in ϵ.

In the first experiment the deflection and torsion were measured only at the end of the beam. The results are given in Table VI and represented in Fig. 7.

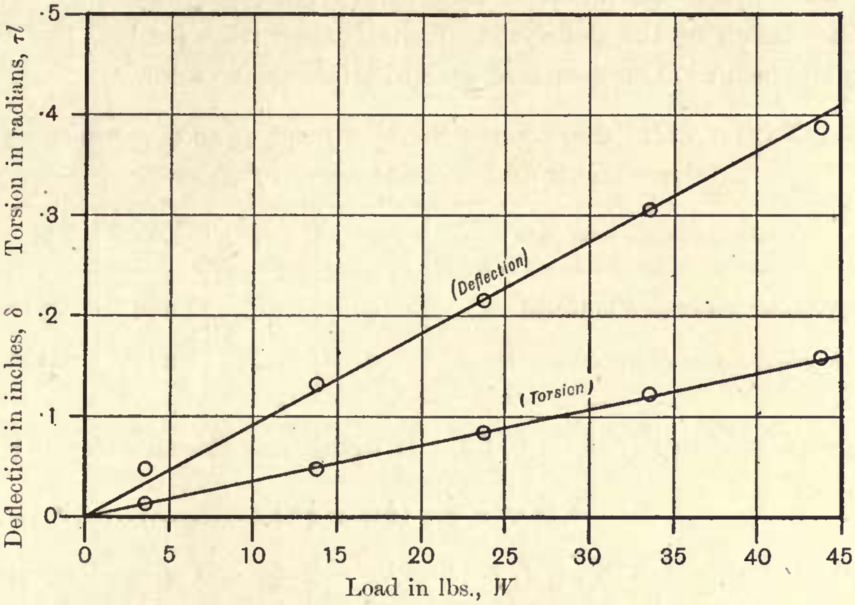


Fig. 7. Deflection and Torsion with varying load of a split pipe*.

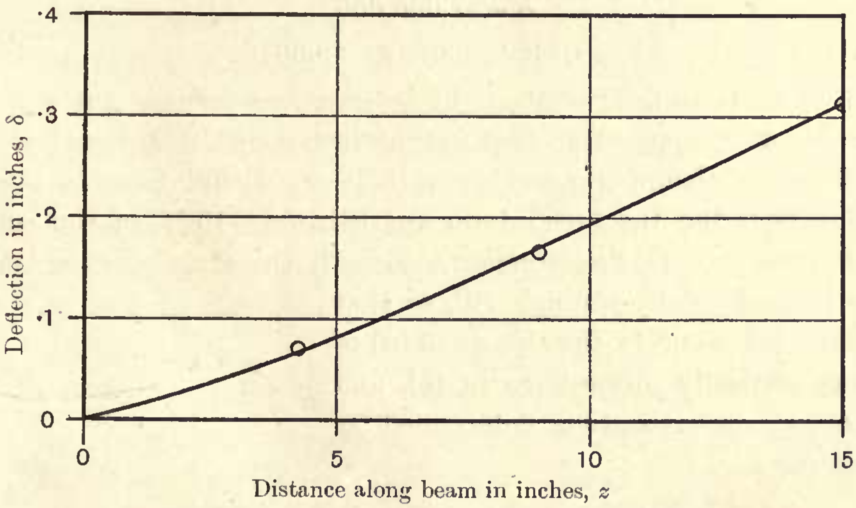


Fig. 8. Curve of Flexure for split pipe.

It is evident that both the deflection and the torsion are proportional to W, but there is a great divergence between the observed values of δ and the values that are given by direct use of the Eulerian expression $\frac{1}{3}\frac{Wl^3}{EI}$ for δ, E being taken to be 27,000,000† lbs. per square inch. This leads us to question whether the linear term s_0z in the expression for the displacement is negligible as it is often taken to be in beams of moderate length. As we shall see the theory does not give much

* The deflections were taken on a 15″, the torsions on a 12″ length, so that the latter need to be multiplied by 1·25.

† This choice is based on calculations given later on in this section.

help because the nature of the clamping is so uncertain, but an experimental test is easy and obvious. If we measure the deflection at various points along the beam we ought to be able to pick out the linear term if it be not negligibly small. Observations were therefore taken of the deflection of the beam with a load of 43·7 lbs. at three points along the beam. The means of several trials are as follows.

TABLE VII. *Experiments with a piece of split piping. Observations to determine curve of flexure.*

Distance along beam (z) in inches:	0	$4\frac{1}{2}$	9	15
Average deflection δ in inches	0	·070	·166	·313*
Approximation of equation (51)......	0	·068	·169	·313

These are plotted in Fig. 8 on p. 39. If we assume the theoretical formula

$$\delta_z = s_0 z + \frac{W}{EI}\left(\tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right) \text{..............................} (51),$$

and substitute the values of W, l, I, we find by the method of least squares that

$$s_0 = {}^{\cdot}014,$$

and

$$E = 27{,}000{,}000,$$

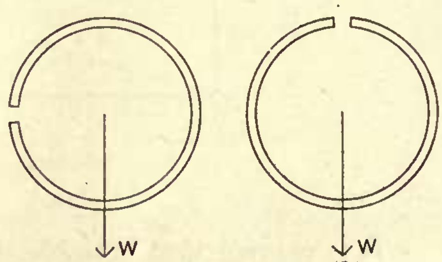
giving

$$\delta_z = {}^{\cdot}014z + {}^{\cdot}00072z^2 - {}^{\cdot}000016z^3.$$

This value of E is quite near what might be expected for this grade of mild steel. The large value of s_0 appears at first extraordinary but it is even less than some theoretical forms of clamping provide; with it we should have to take a length of over 65 inches before the linear term was less than 10 °/₀ of the cubic term in the expression for the end-displacement. Exactly the same experiment was made on the pipe rotated axially through 90°, so that instead of having the cut to the side as in (*a*) of Fig. 9 it was vertically upwards as in (*b*), and the least square approximation determined as before. This gave

$$s_0 = {}^{\cdot}0082,$$
$$E = 27{,}000{,}000,$$

the same value of E as obtained before. Rougher observations for other positions of the cut showed, as one might expect, that the s_0 is a maximum when the cut is at the side and a minimum when the cut is at the top or bottom, these positions giving roughly the same deflection.



(*a*) (*b*)
FIG. 9.

* This small value as compared with that in the previous experiment is, no doubt, due to elastic fatigue. In the first experiment the load was applied by additions of 10 lbs. and between each a considerable time elapsed while the measurements were being made. In the second experiment the 43·7 lbs. load was put on and off at short intervals.

Using the value thus obtained for E we get as our theoretical value for τl, when $W = 43{\cdot}7$ lbs.,

$$\tau l = {\cdot}488.$$

Thus the theoretical values of the torsion are more than double the observed. This may perhaps be accounted for by the roughness of the observations, especially of the thickness of the pipe, for it is evident that, in the neighbourhood of $a_0 = {\cdot}9a$, the value of $\tau \Big/ \frac{Wa}{EI}$ will change with extreme rapidity. For example, if $a_0 = {\cdot}425$, $\tau \Big/ \frac{Wa}{EI} = 214{\cdot}16 + 214{\cdot}18\eta = 267{\cdot}70$ for $\eta = \frac{1}{4}$ and the torsions are then those in the last column of Table VI on p. 38 agreeing almost exactly with the observed. Moreover a tube of this form made out of rolled plate is probably fairly aeolotropic, and this, as will be found, changes the value of τ very largely. There is further the difficulty of arranging the load so that it acts through the centre of gravity, and above all the essential difference of the methods of fixing in practice and in theory.

§ 14. *The nature of the fixing at the end of the beam.*

The experimental results of the preceding paragraph lead us to consider more particularly the nature of the end-conditions—what parts can be fixed and how, so as most nearly to resemble the actual conditions of practice. Physically we see that any extended fixing is impossible with our solution, for, as shown in § 9, it is at the fixed end that the lateral displacements are greatest and to them must be superadded the longitudinal distortion which is the same for every cross-section of the beam. Mathematically we see the impossibility of extended fixing when we consider the fewness of the arbitrary constants which arise in our solution and which we alone have at our disposal to describe the manner of fixing. Bearing our limitations in mind, we proceed to treat some particular cases of theoretical fixings.

From the expressions for the displacements [§ 4, Equation (20)] we derive for the end cross-section, where $z = 0$,

$$\left.\begin{aligned} u &= \frac{W}{EI}\tfrac{1}{2}\eta l\,(x^2 - y^2) - \gamma y + \alpha' \\ v &= \frac{W}{EI}\eta xy\,l + \gamma x + \beta' \\ w &= \chi\,(x, y) - \frac{W}{EI}xy^2 - \beta'' x + \alpha'' y + \gamma' \end{aligned}\right\} \quad \text{..................(52)},$$

and, arranging the differential coefficients in a Jacobian scheme,

$$\left|\begin{matrix} \dfrac{\partial u}{\partial x}, & \dfrac{\partial u}{\partial y}, & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x}, & \dfrac{\partial v}{\partial y}, & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x}, & \dfrac{\partial w}{\partial y}, & \dfrac{\partial w}{\partial z} \end{matrix}\right|_{z=0} \equiv \left|\begin{matrix} \dfrac{W}{EI}\eta lx, & -\dfrac{W}{EI}\eta ly - \gamma, & -\tau y - \dfrac{W}{EI}\tfrac{1}{2}\eta(x^2 - y^2) + \beta \\[2ex] \dfrac{W}{EI}\eta ly + \gamma, & \dfrac{W}{EI}\eta lx, & \tau x - \dfrac{W}{EI}\eta xy - \alpha \\[2ex] \dfrac{\partial \chi}{\partial x} - \dfrac{W}{EI}y^2 - \beta'', & \dfrac{\partial \chi}{\partial y} - \dfrac{2W}{EI}xy + \alpha'', & -\dfrac{W}{EI}xl \end{matrix}\right|$$

.........................(53).

By means of the equations in (52) and (53) we can fix certain points and certain differential coefficients at these or at other points.

(i) *The fixing at one point only, in particular the centroid of the end cross-section.*

Suppose we aim at fixing as far as possible the neighbourhood of the centroid* of the end cross-section, namely $(0, \bar{y}, 0)$. Firstly, to fix the centroid itself, we have from (52),

$$\left.\begin{aligned} u &= -\frac{W}{EI}\tfrac{1}{2}\eta\bar{y}^2 l - \gamma\bar{y} + \alpha' = 0 \\ v &= \beta' = 0 \\ w &= \chi(0, \bar{y}) + \alpha''\bar{y} + \gamma' = 0 \end{aligned}\right\} \text{.........................(54)},$$

and automatically $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial z} = 0$ for any point on the plane $x = 0$, which is the neutral plane.

To determine α we may assume either $\dfrac{\partial v}{\partial z}$ or $\dfrac{\partial w}{\partial y}$ to be zero, since we know that $\alpha = \alpha''$ for our case of symmetry about the axis of y. This makes $\alpha = \alpha'' = 0$, and means that there is to be no rotational displacement of the beam as a whole about the axis of x.

To determine γ we may take $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = 0$, making $\gamma = -\dfrac{W}{EI}\eta l\bar{y}$. This really determines that there is to be no rotational displacement of the beam as a whole about its longitudinal centroidal line.

From equations (54) it follows immediately that $\alpha' = \beta' = \gamma' = 0$.

The only remaining constants are β' and β'', between which there is a relation given by either of equations (37) and (38), so that fixing the one automatically fixes the other. We may either make $\dfrac{\partial u}{\partial z} = 0$ or $\dfrac{\partial w}{\partial z} = 0$.

1°. $\dfrac{\partial u}{\partial z} = 0$. Apart from the mechanical difficulty of introducing a means of fixing which will fix an element in the direction of z, this would make the linear term $(s_0 z)$ in the equation of the curve of flexure vanish, and this is contrary to our experiments. We shall therefore make use of the second alternative which was that adopted by Saint-Venant.

2°. $\dfrac{\partial w}{\partial x} = 0$ or $\left(\dfrac{\partial \chi}{\partial x}\right)_y - \dfrac{W}{EI}\bar{y}^2 - \beta'' = 0.$

This is the same as

$$\left(\frac{\partial \chi}{r\partial\theta}\right)_{\bar{y}} - \frac{W}{EI}\bar{y}^2 - \beta'' = 0,$$

since $\dfrac{\partial}{r\partial\theta} = \dfrac{\partial}{\partial x}$ on the radius of symmetry, or, substituting our value of χ from equation (30),

$$\left[\sum_{i=0}^{\infty}\left(A_m r^{m-1} + \frac{B_m}{r^{m+1}}\right) m\cos m\theta - (\beta - \beta'')\cos\theta + \frac{\tau r}{2\cos\gamma}\, 2\cos 2\theta \right.$$
$$\left. + \frac{W}{3EI}\left[\tfrac{3}{4} + \left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right] 3r^2\cos 3\theta\right]_{\substack{r=\bar{y}\\ \theta=0}} - \frac{W}{EI}\bar{y}^2 - \beta'' = 0,$$

* No property peculiar to the centroid being used, the equations which follow hold good for *any* point whose coordinates are $(0, \bar{y}, 0)$ on the central axis of symmetry.

or

$$\sum_{i=0}^{\infty}\left(A_m\bar{y}^{m-1}+\frac{B_m}{\bar{y}^{m+1}}\right)m-\beta+\frac{\tau\bar{y}}{\cos\gamma}+\frac{W}{EI}\left[\tfrac{3}{4}+\left(\tfrac{1}{4}-\tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right]\bar{y}^2-\frac{W}{EI}\bar{y}^2=0,$$

giving

$$\beta=\sum_{i=0}^{\infty}\left(A_m\bar{y}^{m-1}+\frac{B_m}{\bar{y}^{m+1}}\right)m+\frac{\tau\bar{y}}{\cos\gamma}+\frac{W}{EI}\left[-\tfrac{1}{4}+\left(\tfrac{1}{4}-\tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right]\bar{y}^2 \quad\text{.........(55).}$$

Collecting our assumptions and results, we have:

If the centroid of the end cross-section be fixed so that

$$\left.\begin{aligned}&u=v=w=0\\&\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial w}{\partial z}=0\\&\frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}=0,\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=0,\quad \frac{\partial w}{\partial z}=0\end{aligned}\right\}\text{..................(56),}$$

then the displacement at any point (x, y, z) in the beam is given by

$$\left.\begin{aligned}u=&-\tau yz+\frac{W}{EI}\left[\tfrac{1}{2}\eta\left(x^2-y^2\right)(l-z)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right]+\frac{W}{EI}\eta\, l\bar{y}y\\&+z\left[\sum_{i=0}^{\infty}\left(A_m\bar{y}^{m-1}+\frac{B_m}{\bar{y}^{m+1}}\right)m+\frac{\tau\bar{y}}{\cos\gamma}+\frac{W}{EI}\left(-\tfrac{1}{4}+\left(\tfrac{1}{4}-\tfrac{1}{2}\eta\right)\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right)\bar{y}^2\right]\\v=&\,\tau zx+\frac{W}{EI}\eta xy(l-z)-\frac{W}{EI}\eta l\bar{y}x\\w=&\,\chi(x,y)-\frac{W}{EI}\left(x\left(lz-\tfrac{1}{2}z^2\right)+xy^2\right)-\left(\left(\frac{\partial\chi}{\partial x}\right)_{\bar{y}}-\frac{W}{EI}\bar{y}^2\right)x\end{aligned}\right\}\text{...(57).}$$

If we consider in particular the curve formed by the line of centroids, that is the curve obtained by putting $x=0$, $y=\bar{y}$ in the equations of (57), we obtain after a little reduction

$$\left.\begin{aligned}u(\bar{y})=&\frac{W}{EI}\left(\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right)+\frac{Wa^2}{EI}\left[\sum_{i=0}^{\infty}\left(A_m\bar{y}^{m-1}+\frac{B_m}{\bar{y}^{m+1}}\right)m\right.\\&\left.+\tau\Big/\frac{Wa}{EI}\cdot\frac{\bar{y}}{a}\left(\frac{1}{\cos\gamma}-1\right)+\frac{\bar{y}^2}{a^2}\left(-\frac{1}{8}+\frac{1}{8}\frac{\cos\frac{1}{2}\gamma}{\cos\frac{3}{2}\gamma}\right)\right]z\end{aligned}\right.$$

assuming uni-constant isotropy,

$$=\frac{W}{EI}\left(\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right)+\frac{\widehat{zx}_{\bar{y}}}{\mu}z$$

where $\widehat{zx}_{\bar{y}}$ is the vertical shearing stress at the centroid of any cross-section (from equation 45), and

$$v(\bar{y})=0,\qquad w(\bar{y})=0. \quad\text{...(58).}$$

(*a*) *Case of the complete sector.*

For this case the distribution* of the vertical shear along the radius of symmetry has been worked out, and we can determine the values of $\widehat{zx}$ at the centroid or at any

* See § 12, Table III and Fig. 4.

other point with sufficient accuracy by interpolation. We thus have the following table for the value of the coefficient of the linear term in $u(\bar{y})$.

TABLE VIII.

Angle of sector:	5°	12°	30°	45°	60°	72°	90°	120°	180°	270°	360°
$\dfrac{\widehat{zx_{\bar{y}}}}{\mu} = \dfrac{Wa^2}{EI} \times$	·00115	·00674	·04144	·08896	·15127	·20726	·30371	·49983	·89385	1·20921	∞

It might be anticipated that a better approximation to what would arise in practical fixings, when the sector has a small angle, would be obtained by taking the theoretical point of fixing to lie between the centroid and the circumference end of the radius of symmetry. The nature of the corresponding values of $\dfrac{\widehat{zx}}{\mu}$ can be immediately seen from Table III. Such a point of fixing would enable us to avoid the infinite linear term in the deflection for the split cylinder.

(*b*) *Case of the sector, complete or curtate, when* $\gamma = 2\pi$.

In this case $\bar{y}$ is at the origin and when $\bar{y} = 0$, $\widehat{zx_{\bar{y}}}$ becomes *infinite* for the complete sector. Moreover we must always have the relation of (38) satisfied whatever be our theoretical method of fixing and it is clear that for our case of $\gamma = 2\pi$ the value of $\beta - \beta''$ will have to be infinite, since τ is not zero.

These mathematical infinities need not deter us from applying the theory to physical problems. In the actual experiments we cannot get angles of exactly 2π and we have no means of practical fixing in accordance with the theoretical methods. In fact the true interpretation of these infinities is that we need not be surprised at relatively large numbers arising at the corresponding points in physical problems and it was the large value found in the experiments with the cut pipe which led us to the investigations in this section.

(ii) *Fixing at two points—case of cylindrical split pipe.*

Instead of fixing one point and certain directions through that point we may use the arbitrary constants in the solution to fix two points of the terminal face. The choice of the two points would naturally vary with different kinds of sector and we shall devote our attention immediately to the cylindrical split pipe.

We shall choose the two points $(\pm a, 0, 0)$, i.e. the two points where the plane of flexure crosses the outer circle of the end-section. The equations for $u = v = w = 0$ will then be

$$\left.\begin{aligned} u &= \frac{W}{EI}\tfrac{1}{2}\eta a^2 l \qquad\qquad + \alpha' = 0 \\ v &= \qquad\qquad \pm\gamma a + \beta' = 0 \\ w &= \chi\begin{pmatrix} r = a \\ \theta = \pm\tfrac{1}{2}\pi \end{pmatrix} \mp \beta'' a + \gamma' = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(59).$$

From these we deduce immediately

$$\alpha' = -\frac{W}{EI}\tfrac{1}{2}\eta a^2 l, \quad \beta' = 0, \quad \gamma' = 0, \quad \gamma = 0,$$

and
$$\beta'' a = \chi\left(a, \frac{\pi}{2}\right).$$

To determine α we may put $\frac{\partial w}{\partial y} = 0$ at $(\pm a, 0, 0)$—making $\alpha = 0$ and virtually preventing a rotation about the diameter through $(\pm a, 0, 0)$.

The equation for β'' is from equation (30)

$$\beta'' a = \sum_{i=0}^{\infty}\left(A_m a^m + \frac{B_m}{a^m}\right)\sin\frac{m\pi}{2} - (\beta - \beta'')\, a - \frac{W}{3EI}\left[\tfrac{3}{4} + \left(\tfrac{1}{4} - \tfrac{1}{2}\eta\right)\right] a^3,$$

whence
$$\beta = \sum_{i=0}^{\infty}\left(A_m a^{m-1} + \frac{B_m}{a^{m+1}}\right)\sin(2i+1)\frac{\pi}{4} - \frac{W}{3EI}\left(1 - \tfrac{1}{2}\eta\right) a^2 \quad \text{.........(60)},$$

or, substituting for A_m and B_m from (34),

$$\beta = \sum_{i=0}^{\infty}\frac{W}{EI}\frac{a^2(-1)^i}{2\pi}\left[-\frac{8\times 520{\cdot}6\,(1+\eta)}{m\,(m^2-4)}\left(\frac{1 - 2\left(\frac{a_0}{a}\right)^{m+2} + \left(\frac{a_0}{a}\right)^{2m}}{1 - \left(\frac{a_0}{a}\right)^{2m}}\right)\right.$$
$$\left. - \left(\frac{3+2\eta}{m\,(m^2-1)} - \frac{3-6\eta}{m\,(m^2-9)}\right)\left(\frac{1 - 2\left(\frac{a_0}{a}\right)^{m+3} + \left(\frac{a_0}{a}\right)^{2m}}{1 - \left(\frac{a_0}{a}\right)^{2m}}\right)\right]\sin(2i+1)\frac{\pi}{4}$$
$$-\frac{1}{3}\frac{W}{EI}\left(1 - \tfrac{1}{2}\eta\right) a^2 \quad \text{.....................................(61)}.$$

With these determinations of the arbitrary constants the displacement equations become

$$\left.\begin{aligned} u &= -\tau yz + \frac{W}{EI}\left[\tfrac{1}{2}\eta\,(x^2 - y^2)(l - z) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right] + \beta z - \frac{W}{EI}\tfrac{1}{2}\eta a^2 l \\ v &= \tau zx + \frac{W}{EI}\eta xy\,(l - z) \\ w &= \chi(x, y) - \frac{W}{EI}\left[x\left(lz - \tfrac{1}{2}z^2\right) + xy^2\right] - \frac{1}{a}\chi\left(a, \frac{\pi}{2}\right) \end{aligned}\right\} \quad \text{...(62)},$$

and in particular the curve of centroids, which for the sector in which $\gamma = 2\pi$ is the same as the curve of centres, is

$$u_0 = \frac{W}{EI}\left[\tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right] + \beta z - \frac{W}{EI}\tfrac{1}{2}\eta a^2 l \quad \text{..................(63)}.$$

The last constant term is of course the result of the anticlastic curvature and as we shall show is of no importance compared with the value of βl. The value of β is identical with s_0 as in all cases where the centroid coincides with the centre.

Returning, therefore, to our value of β in equation (61) we find that the first ten terms of the series are needful to give adequate accuracy when $a_0 = \frac{9}{10}a$, but the only terms that are of real importance are those containing the large constant $520\cdot6\,(1+\eta)$. The result of the calculation is that

$$\beta = s_0 = \frac{Wa^2}{EI} \times 927\cdot931 \quad\text{.......................... (64)},$$

an extraordinary value when compared with the values (for angles other than 360°) in Table VIII, and with the comparable values for the case of the beam of rectangular cross-section calculated by Saint-Venant*.

This large value is, however, confirmed by our experiment described in § 13. For when we calculate β for a pipe of the dimensions used in the experiment, we find that $\beta = \cdot0240$. This is somewhat larger than the observed value which was ·014. However it is of the same order when we compare it with other cases.

It is interesting, for instance, to compare it with the value of β deduced by the same method of fixing if the pipe be closed. It is easy to show for this case of the complete hollow circular cylindrical pipe that

$$\chi = \frac{W}{EI}\left(\tfrac{3}{4}+\tfrac{1}{2}\eta\right)\left((a^2+a_0^2)\,r + \frac{a^2 a_0^2}{r}\right)\sin\theta + \frac{W}{EI}\,\tfrac{1}{4}r^3\sin 3\theta - (\beta-\beta'')\,r\sin\theta \quad\text{...(65)},$$

the notation being the same as used for the curtate cylindrical sector, and that the value of β, when the points $r=a$, $\theta = \pm\frac{\pi}{2}$, $z=0$ are fixed, is given by

$$\beta = \frac{W}{EI}\left(\tfrac{5}{8}a^2 + \tfrac{7}{4}a_0^2\right) \quad\text{............................. (66)},$$

or, when $a = 0\cdot50$ inches and $a_0 = 0\cdot45$ inches,

$$\beta = \cdot000053,$$

—about $\frac{1}{500}$th of the corresponding β when there is a cut in the pipe.

Now the plugging of a split pipe and the clamping of the plugged end in the chuck of a lathe may not very closely correspond to fixing indefinitely small elements at the top and bottom of the external circumference at the fixed end of the pipe. It is worth while inquiring what value β would take, if we fixed external elements on the horizontal diameter at the clamped end. We may take as our conditions:

$$u,\ v,\ w = 0 \text{ for } z = x = 0,\ r = a,\ \theta = 0,$$

and again for

$$r = a,\ \theta = \pi.$$

From (35) we have

$$u = 0 = \frac{W}{EI}\left(-\tfrac{1}{2}\eta a^2\right) l \pm \gamma a + \alpha',$$

whence flow

$$\gamma = 0,\ \alpha' = \tfrac{1}{2}\frac{Wa^2}{EI}\eta l,\ v = 0,\ \beta' = 0;$$

$$w = 0 = \chi(a, 0) + \alpha'' a + \gamma',$$

$$w = 0 = \chi(a, \pi) - \alpha'' a + \gamma'.$$

* "Memoir on Flexure," § 29 (*Journal de Liouville*, Series II, Tome I (1856)). Quoted in Todhunter and Pearson's *History*, Vol. II, Section [96].

Now $\chi(a, 0)$ is zero, and

$$\chi(a, \pi) = \sum_{i=0}^{\infty} \left(A_m a^m + \frac{B_m}{a^m} \right) \sin(2i+1)\frac{\pi}{2}$$

does not involve any additional unknown constant.

Hence these equations determine

$$\alpha'' = \frac{1}{2a}\chi(a, \pi) \text{ and } \gamma' = -\tfrac{1}{2}\chi(a, \pi).$$

It still remains to select a condition which provides β. A suitable limitation seems to be that dw/dx should be zero at non-split edge, or

$$0 = \frac{dw}{dx} = \left(\frac{d\chi}{rd\theta}\right)_{\substack{r=a\\ \theta=0}} - \frac{W}{EI}a^2 - \beta'',$$

which leads to
$$\beta = \sum_{i=0}^{\infty} \left(A_m a^{m-1} + \frac{B_m}{a^{m+1}} \right) m + \tau a - \tfrac{1}{2}\eta \frac{Wa^2}{EI}.$$

Substituting for A_m and B_m from (34) and the value $\frac{Wa}{EI} \times 520{\cdot}6(1+\eta)$ for τ, we have

$$\beta = \frac{Wa^2}{EI}\left\{ \sum_0^{\infty} \left[\frac{(-1)^i}{2\pi} \left\{ \frac{-8 \times 520{\cdot}6(1+\eta)}{m^2-4} \frac{1 - 2\left(\frac{a_0}{a}\right)^{m+2} + \left(\frac{a_0}{a}\right)^{2m}}{1 - \left(\frac{a_0}{a}\right)^{2m}} \right.\right.\right.$$
$$\left.\left.\left. - \left(\frac{3+2\eta}{m^2-1} - \frac{3-6\eta}{m^2-9}\right) \frac{1 - 2\left(\frac{a_0}{a}\right)^{m+3} + \left(\frac{a_0}{a}\right)^{2m}}{1 - \left(\frac{a_0}{a}\right)^{2m}} \right\} \right] + 520{\cdot}6(1+\eta) - \tfrac{1}{2}\eta \right\},$$

or, if $\eta = \frac{1}{4}$,
$$\beta = \frac{Wa^2}{EI} \times 959{\cdot}675,$$

a result comparing fairly with the previous high value 927·931 obtained by fixing elements on the vertical diameter. It seems necessary to infer that with perfect clamping we should reach a value of β comparable with that given in (64).

Experimental verifications of the foregoing theory.

It will be of advantage at this stage to summarise some of the results of experiments made with a view to increasing the faith in the theory. In all the cases considered here, the deflection (δ_z) was observed at various points (z) along the lowest generator* of the cylinder, and the constants s_0 and s_1 in the formula

$$\delta_z = s_0 z + s_1 (\tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3) \qquad \text{(67)}$$

were deduced from the measurements by the method of least squares. The results are given in Table IX. Remembering that s_0 seems naturally to arise in the form

$$s_0 = (Wa^2/EI) \times \text{a quantity depending on the shape of the cross-section,}$$

* s_0 will therefore differ from β by a term of the form $\frac{Wa^2}{EI}\frac{1}{2}\eta$. This term would have no sensible effect in the case of the split pipes, but would mean that s_0 is less than β by about 6 °/。 in the case of the non-split tube.

where a is half the typical vertical dimension of the cross-section of the beam*, we shall write, after Saint-Venant, the *total* deflection as

$$\delta_t = \frac{W}{3EI} l^3 \left(1 + \frac{3a^2}{l^2} \sigma_0\right) \text{ ....(68).}$$

The quantity σ_0 is tabulated as well as the ratio s_0/s_1. For purposes of comparison, of course, σ_0 is superior to s_0/s_1.

TABLE IX. *Experimental results on the nature of the deflection with various beams:* $\delta_z = s_0 z + s_1 (\frac{1}{2} l z^2 - \frac{1}{6} z^3)$.

Description and dimensions of beam			s_0 (β)	s_1	$\frac{s_0}{s_1}$	$\sigma_0 = \frac{s_0/s_1}{a^2}$	Stretch-modulus E lbs. per sq. in.
Split pipe (soft steel): cut at side		$a = 0{\cdot}50$ in. $a_0 = 0{\cdot}45$ in. $l = 15$ in.	·0136 (·0240)	·000096	140	560	27,000,000
Same pipe: cut on top		$a = 0{\cdot}50$ in. $a_0 = 0{\cdot}45$ in. $l = 15$ in.	·0082 (·0068)	·000096	85	340	27,000,000
Soft steel pipe without cut		$a = \frac{3}{8}$ in. $a_0 = \frac{5}{16}$ in. $l = 9{\cdot}5$ in.	·00181 (·000061)	·000236	7·7	54	23,000,000
Bass-wood cylinder with saw-cut to central axis		$a = \frac{5}{8}$ in. $l = 32$ in.	·0151 (·00046)	·0000681	220	570	1,470,000

The experiments were only rough and little was expected of them beyond some slight verification of the theory. The values, however, of the Young's Modulus, E, are in accordance with what we should anticipate for the materials used. The chief divergence lies in the values of s_0 when compared with the theoretical values of β in the case of the solid pipe and the wooden cylinder with the saw-cut. But the nature of clamping, by a chuck for the uncut pipe and a simple clamp for the bass-wood cylinder, is really not that of two fixed points and may well provide somewhat more closely the high values of a central fixing: see (*b*), p. 44.

We may note: 1°. the difference between the σ_0's for the split pipe when the cut is at the side and when it is on the top (the arrows in the drawings of the cross-sections being in the direction of the flexure);

2°. the small value of σ_0 for the complete tube although it is not so small as the method of fixing by the two points $(\pm a, 0, 0)$ described above would provide;

3°. the σ_0 for the wooden beam with the saw-cut up to the central axis. This is the physical approximation to the non-curtate sector of angle 2π. The σ_0 is large as we should expect from a consideration of the order of the ratio of stretch-modulus (E) to slide-modulus (μ) for wood. This is considered more fully in the next section of the paper.

* In our case the outer radius of the sector, in the case of a beam of rectangular section the semi-thickness, and for an elliptic section the semi-axis parallel to the direction of flexure.

IV. Solution for certain Aeolotropic Cases.

§ 15. *A solution for aeolotropic material.*

Reverting to our original equations in § 3 for a uniformly aeolotropic substance, we see that the main change in the analysis that a consideration of aeolotropy requires will be introduced by the equation (15) of p. 10

$$\mu_1 \frac{\partial^2 \chi}{\partial x^2} + \mu_2 \frac{\partial^2 \chi}{\partial y^2} = 0 \quad \text{..........................(69 } a\text{)},$$

instead of the usual Laplace equation. It is natural to try a projection, which will transform this equation into Laplace's equation. Let us project, then, with the substitution

$$x = \sqrt{\frac{\mu_1}{\mu_2}}\, x', \qquad y = y' \quad \text{..........................(69 } b\text{)}.$$

This projection depends on the nature of the aeolotropy, circles being changed into ellipses of semi-axes proportional to $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$, but in such a way that there is *no change in the lateral dimension* of the beam.

If the boundary be $\phi(x, y) = 0$,

the transformed boundary will be

$$\phi\left(\sqrt{\frac{\mu_1}{\mu_2}}\, x', y'\right) = 0,$$

or

$$\phi'(x', y') = 0,$$

if we agree to denote everything in connection with the transformed problem with dashed letters. Also

$$\cos(x'\nu') = \frac{\dfrac{\partial \phi'}{\partial x'}}{\sqrt{\left\{\left(\dfrac{\partial \phi'}{\partial x'}\right)^2 + \left(\dfrac{\partial \phi'}{\partial y'}\right)^2\right\}}} = \frac{\sqrt{\dfrac{\mu_1}{\mu_2}}\dfrac{\partial \phi}{\partial x}}{\sqrt{\left\{\dfrac{\mu_1}{\mu_2}\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2\right\}}}$$

$$= \sqrt{\frac{\mu_1}{\mu_2}} \cos(x\nu) \times \sqrt{\frac{\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2}{\dfrac{\mu_1}{\mu_2}\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2}} \quad \text{..................(70)},$$

and, similarly,

$$\cos(y'\nu') = \cos(y\nu) \times \sqrt{\frac{\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2}{\dfrac{\mu_1}{\mu_2}\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2}} \quad \text{........................(71)}.$$

Thus the problem of the aeolotropic case, namely:

To find $\chi(x, y)$ and τ to satisfy

$$\mu_1 \frac{\partial^2 \chi}{\partial x^2} + \mu_2 \frac{\partial^2 \chi}{\partial y^2} = 0$$

with the boundary condition

$$\mu_1 \frac{\partial\chi}{\partial x}\cos(x\nu)+\mu_2\frac{\partial\chi}{\partial y}\cos(y\nu)=\tau\left(\mu_1 y\cos(x\nu)-\mu_2 x\cos(y\nu)\right)$$
$$+\frac{W}{EI}\left[\left(\tfrac{1}{2}\eta_1\mu_1x^2+\frac{(E-\eta_1\mu_1-2\eta_2\mu_2)\,\mu_1}{2\mu_2}y^2\right)\cos(x\nu)+(E-\eta_1\mu_1)\,xy\cos(y\nu)\right]$$
$$-\mu_1(\beta-\beta'')\cos(x\nu)+\mu_2(\alpha-\alpha'')\cos(y\nu) \quad \text{..............................(72)},$$

and the Conditions I and II,

$$0=\int\chi\cos(y\nu)\,ds \quad \text{..............................(73)},$$

$$-W\bar{y}=\iint\left[\tau(\mu_2x^2+\mu_1y^2)+\left(\mu_2x\frac{\partial\chi}{\partial y}-\mu_1y\frac{\partial\chi}{\partial x}\right)-\frac{W}{EI}(E-\tfrac{3}{2}\eta_1\mu_1)\,x^2y\right.$$
$$\left.+\frac{W}{EI}\frac{(E-\eta_1\mu_1-2\eta_2\mu_2)\,\mu_1}{2\mu_2}y^3-\mu_2(\alpha-\alpha'')\,x-\mu_1(\beta-\beta'')\,y\right]dxdy\text{...(74)},$$

is projected into the problem:

To find $\chi'(x', y')$ and τ' to satisfy

$$\frac{\partial^2\chi'}{\partial x'^2}+\frac{\partial^2\chi'}{\partial y'^2}=0$$

with the boundary condition

$$\frac{\partial\chi'}{\partial x'}\cos(x'\nu')+\frac{\partial\chi'}{\partial y'}\cos(y'\nu')=\tau'\sqrt{\frac{\mu_1}{\mu_2}}\left(y'\cos(x'\nu')-x'\cos(y'\nu')\right)$$
$$+\frac{W}{EI}\sqrt{\frac{\mu_1}{\mu_2}}\left[\left(\frac{\eta_1\mu_1}{2\mu_2}x'^2+\frac{E-\eta_1\mu_1-2\eta_2\mu_2}{2\mu_2}y'^2\right)\cos(x'\nu')+\frac{E-\eta_1\mu_1}{\mu_2}x'y'\cos(y'\nu')\right]$$
$$-(\beta-\beta'')'\sqrt{\frac{\mu_1}{\mu_2}}\cos(x'\nu')+(\alpha-\alpha'')'\cos(y'\nu') \quad \text{..............................(75)},$$

and the Conditions I and II,

$$0=\int\chi'\cos(y'\nu')\,ds' \quad \text{..............................(76)},$$

$$-W\bar{y}'=\iint\left[\tau'\mu_1(x'^2+y'^2)+\sqrt{\mu_1\mu_2}\left(x'\frac{\partial\chi'}{\partial y'}-y'\frac{\partial\chi'}{\partial x'}\right)\right.$$
$$-\frac{W}{EI}(E-\tfrac{3}{2}\eta_1\mu_1)\frac{\mu_1}{\mu_2}x'^2y'+\frac{W}{EI'}\frac{(E-\eta_1\mu_1-2\eta_2\mu_2)\,\mu_1}{2\mu_2}y'^3$$
$$\left.-(\alpha-\alpha'')'\sqrt{\mu_1\mu_2}\,x'-\mu_1(\beta-\beta'')'\,y'\right]\sqrt{\frac{\mu_1}{\mu_2}}\,dx'dy' \quad \text{..............(77)}.$$

This projected problem is analytically exactly the same as for the isotropic case*. The only difference from the latter lies in the different constants in the three condition-equations. Hence any solution that has been obtained for an isotropic case can be used to obtain a solution for a connected problem in an aeolotropic material. There is however the limitation which must be emphasised (to the depreciation of the method), that the form of the soluble cases depends essentially on the solubility

* Note that $I=(\mu_1/\mu_2)^{\frac{3}{2}}I'$ has been retained and is therefore $=\frac{1}{8}(\mu_1/\mu_2)^{\frac{3}{2}}(a^4-a_0^4)(\gamma'-\sin\gamma')$ for the 'curtate elliptical sector.'

of a corresponding isotropic case and also on the nature of the aeolotropy—to be precise, on the ratio of the two slide-moduli μ_1, μ_2. The result is that the soluble cases cannot always be for cross-sections of common occurrence, but must be for distortions of such. In the uncertainty of the elastic constants of aeolotropy, however, this disadvantage may in many cases be of little importance.

§ 16. *Case of the curtate sector cross-section.*

Let us take the *projected* figure to be the curtate circular sector which is enclosed between the circles

$$\left.\begin{aligned} x'^2+y'^2&=a^2\\ x'^2+y'^2&=a_0^2\\ \text{and the lines}\qquad y'&=\pm x'\tan\tfrac{1}{2}\gamma' \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(78).$$

Then on projecting back by the transformation

$$x'=\sqrt{\frac{\mu_2}{\mu_1}}\,x,\qquad y'=y,$$

the cross-section given will be a 'curtate elliptical sector' enclosed between the ellipses

$$\left.\begin{aligned} \frac{\mu_2}{\mu_1}x^2+y^2&=a^2\\ \frac{\mu_2}{\mu_1}x^2+y^2&=a_0^2\\ \text{and the lines}\qquad y=\pm\sqrt{\frac{\mu_2}{\mu_1}}\,x\tan\tfrac{1}{2}\gamma'&=\pm x\tan\tfrac{1}{2}\gamma,\ \text{say} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(79).$$

Following the method of treatment used for the isotropic case in § 6, we get the following solution for χ' in the projected problem:

$$\chi'=\sum_{i=0}^{\infty}\left(A'_m r'^m+\frac{B'_m}{r'^m}\right)\sin m\theta'-(\beta-\beta'')'\sqrt{\frac{\mu_1}{\mu_2}}\,r'\sin\theta'+\tau'\sqrt{\frac{\mu_1}{\mu_2}}\frac{r'^2}{2\cos\gamma'}\sin 2\theta'$$
$$+\frac{1}{3}\frac{W}{EI}\sqrt{\frac{\mu_1}{\mu_2}}\left[\frac{\frac{3}{8}E-\frac{1}{2}\eta_1\mu_1-\frac{1}{4}\eta_2\mu_2}{\mu_2}+\frac{\frac{1}{8}E-\frac{3}{4}\eta_2\mu_2}{\mu_2}\frac{\cos\frac{1}{2}\gamma'}{\cos\frac{3}{2}\gamma'}\right]r'^3\sin 3\theta'\ldots\ldots(80),$$

where

$$\left.\begin{aligned} A'_m={}&-8\tau'\sqrt{\frac{\mu_1}{\mu_2}}\frac{(-1)^i}{(m^2-4)\,m\gamma'}\frac{a^{m+2}-a_0^{m+2}}{a^{2m}-a_0^{2m}}\\ &+(-1)^i\frac{W}{EI}\sqrt{\frac{\mu_1}{\mu_2}}\frac{\cos\frac{1}{2}\gamma'}{m\gamma'}\left(\frac{\frac{3}{2}E-\eta_2\mu_2}{\mu_2}\frac{1}{m^2-1}-\frac{\frac{3}{2}E-9\eta_2\mu_2}{\mu_2}\frac{1}{m^2-9}\right)\frac{a^{m+3}-a_0^{m+3}}{a^{2m}-a_0^{2m}}\\ B'_m={}&8\tau'\sqrt{\frac{\mu_1}{\mu_2}}\frac{(-1)^i}{(m^2-4)\,m\gamma'}\frac{a^{m+2}a_0^{m+2}\left(a^{m-2}-a_0^{m-2}\right)}{a^{2m}-a_0^{2m}}\\ &-(-1)^i\frac{W}{EI}\sqrt{\frac{\mu_1}{\mu_2}}\frac{\cos\frac{1}{2}\gamma'}{m\gamma'}\left(\frac{\frac{3}{2}E-\eta_2\mu_2}{\mu_2}\frac{1}{m^2-1}-\frac{\frac{3}{2}E-9\eta_2\mu_2}{\mu_2}\frac{1}{m^2-9}\right)\\ &\qquad\times\frac{a^{m+3}a_0^{m+3}\left(a^{m-3}-a_0^{m-3}\right)}{a^{2m}-a_0^{2m}}\\ \text{and}\qquad m={}&(2i+1)\frac{\pi}{\gamma'},\qquad i=0,\ 1,\ 2,\ \ldots \end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(81).$$

Condition I will as before (p. 17) be identically satisfied for symmetry perpendicular to the plane of loading applying Condition II, and remembering the value of I (see ftn. p. 50), we get as equation for the torsion τ',

$$\tau'\Bigg[(a^4-a_0^4)(\gamma'-\tan\gamma')+\sum_{i=0}^{\infty}\frac{64}{(m^2-4)(m+2)m\gamma'}\frac{(a^{m+2}-a_0^{m+2})^2}{a^{2m}-a_0^{2m}}$$
$$-\sum_{i=0}^{\infty}\frac{64}{(m^2-4)(m-2)m\gamma'}\frac{a^4a_0^4(a^{m-2}-a_0^{m-2})^2}{a^{2m}-a_0^{2m}}\Bigg]$$
$$=\frac{W}{EI}\Bigg[\sum_{i=0}^{\infty}\frac{2\cos\frac{1}{2}\gamma'}{(m+2)m\gamma'}\left(\frac{6E-4\eta_2\mu_2}{\mu_2}\frac{1}{m^2-1}-\frac{6E-36\eta_2\mu_2}{\mu_2}\frac{1}{m^2-9}\right)$$
$$\times\frac{(a^{m+2}-a_0^{m+2})(a^{m+3}-a_0^{m+3})}{a^{2m}-a_0^{2m}}$$
$$-\sum_{i=0}^{\infty}\frac{2\cos\frac{1}{2}\gamma'}{(m-2)m\gamma'}\left(\frac{6E-4\eta_2\mu_2}{\mu_2}\frac{1}{m^2-1}-\frac{6E-36\eta_2\mu_2}{\mu_2}\frac{1}{m^2-9}\right)$$
$$\times\frac{a^5a_0^5(a^{m-3}-a_0^{m-3})(a^{m-2}-a_0^{m-2})}{a^{2m}-a_0^{2m}}$$
$$+\frac{2}{15}\frac{E-6\eta_2\mu_2}{2\mu_2}\cos\tfrac{1}{2}\gamma'\tan\tfrac{3}{2}\gamma'\,(a^5-a_0^5)$$
$$-\frac{2}{5}\frac{E-6\eta_2\mu_2}{2\mu_2}\sin\tfrac{1}{2}\gamma'\,(a^5-a_0^5)$$
$$-\frac{2}{3}\frac{E}{\mu_2}\left(1-\frac{\sin\gamma'}{\gamma'}\right)\sin\tfrac{1}{2}\gamma'\,\frac{(a^3-a_0^3)(a^4-a_0^4)}{a^2-a_0^2}\Bigg]\quad\ldots\ldots\ldots\ldots(82).$$

Now projecting back, so far as this equation is concerned, consists merely in replacing* τ' by τ, and γ' by γ, where $\gamma'=2\tan^{-1}\left(\sqrt{\frac{\mu_1}{\mu_2}}\tan\frac{1}{2}\gamma\right)$ and retaining I, so that with these changes the equation (82) gives the value of the torsion accompanying the flexure of a beam having the curtate elliptical sector cross-section given by the equations (79).

§ 17. *Solution in the case of transverse isotropy.*

Transverse isotropy consists in the slide-modulus and the Poisson ratio being independent of the direction perpendicular to the longitudinal axis of the beam. We may write $\mu_1=\mu_2=\mu$ and $\eta_1=\eta_2=\eta$, but we do not necessarily have $E=2\mu(1+\eta)$ which would give the case of complete isotropy.

Applying the method of the previous section we see that the projected figure is the same as the original figure in this case, but we can deduce the result from the aeolotropic case and have immediately from equation (82),

* Of course 'replacing' τ' by τ, and retaining I does not involve *equality* of τ' and τ or of I' and I.

$$\tau\left[(a^4-a_0^4)(\gamma-\tan\gamma)+\sum_{i=0}^{\infty}\frac{64}{(m^2-4)(m+2)m\gamma}\frac{(a^{m+2}-a_0^{m+2})^2}{a^{2m}-a_0^{2m}}\right.$$

$$\left.-\sum_{i=0}^{\infty}\frac{64}{(m^2-4)(m-2)m\gamma}\frac{a^4a_0^4(a^{m-2}-a_0^{m-2})^2}{a^{2m}-a_0^{2m}}\right]$$

$$=\frac{W}{EI}\left[\sum_{i=0}^{\infty}\frac{2\cos\frac{1}{2}\gamma}{(m+2)m\gamma}\left(\left(6\frac{E}{\mu}-4\eta\right)\frac{1}{m^2-1}-\left(6\frac{E}{\mu}-36\eta\right)\frac{1}{m^2-9}\right)\right.$$

$$\times\frac{(a^{m+2}-a_0^{m+2})(a^{m+3}-a_0^{m+3})}{a^{2m}-a_0^{2m}}$$

$$-\sum_{i=0}\frac{2\cos\frac{1}{2}\gamma}{(m-2)m\gamma}\left(\left(6\frac{E}{\mu}-4\eta\right)\frac{1}{m^2-1}-\left(6\frac{E}{\mu}-36\eta\right)\frac{1}{m^2-9}\right)$$

$$\times\frac{a^5a_0^5(a^{m-3}-a_0^{m-3})(a^{m-2}-a_0^{m-2})}{a^{2m}-a_0^{2m}}$$

$$+\frac{2}{15}\left(\frac{1}{2}\frac{E}{\mu}-3\eta\right)\cos\tfrac{1}{2}\gamma\tan\tfrac{3}{2}\gamma\,(a^5-a_0^5)$$

$$-\frac{2}{5}\left(\frac{1}{2}\frac{E}{\mu}-3\eta\right)\sin\tfrac{1}{2}\gamma\,(a^5-a_0^5)$$

$$\left.-\frac{2}{3}\frac{E}{\mu}\left(1-\frac{\sin\gamma}{\gamma}\right)\sin\tfrac{1}{2}\gamma\,\frac{(a^3-a_0^3)(a^4-a_0^4)}{a^2-a_0^2}\right]\quad\ldots\ldots(83).$$

A comparison with (82) shows us that this result is absolutely the same if the appropriate E/μ_2 and η_2 be used. In other words the difference of the two cases lies solely in the distortion of the auxiliary cross-section.

§ 18. *Note on the torsion of an aeolotropic beam with flexure.* Cf. § 9.

The equation for the vertical displacement at the point (x, y, z) in the aeolotropic case is (equation 14)

$$u=-\tau yz+\frac{W}{EI}\left[\tfrac{1}{2}(l-z)(\eta_1x^2-\eta_2y^2)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right]-\gamma y+\beta z+\alpha',$$

the only change from the isotropic case being the introduction of the two Poisson's ratios η_1 and η_2. When we consider the displacements on the horizontal axis of symmetry the only Poisson's ratio occurring is the η_2, and we have as the torsion in the case of the beam discussed in the previous section

$$\psi_l-\psi_0=-\tau l+\frac{Wl\eta_2}{2EI}(a+a_0)\quad\ldots\ldots(84).$$

Compare equation (42).

Thus the torsion due to the anticlastic curvature depends only on that Poisson's ratio which is the contraction in the direction of y (i.e. perpendicular to the plane of flexure) due to an extension in the direction of z, the longitudinal axis.

§ 19. *Applications to certain aeolotropic materials and experimental results.*

The interest in the aeolotropic case does not lie so much in the immediate usefulness of the formulae as in the estimation of the order of magnitude of the changes that the aeolotropy of a commonly occurring kind is likely to produce. Aeolotropic materials other than those of regular crystalline structure are by nature so variable in their aeolotropy that it is impossible to give dependable values to their elastic constants. In the case of the common woods for instance all depends on the part of the tree trunk that is selected, and in any one plank the aeolotropy may be so variable as to upset even the nature of the mathematical result proved for uniform aeolotropy. Thus in an experiment that was made with a plank about four inches broad and bevelled so as to have for cross-section approximately a circular sector of small angle, it was found that the torsion caused by flexure was most pronounced but in the direction *contrary* to that given by theory. The annual rings in the wood were noticed to be much wider apart towards one edge of the plank and, this explanation having suggested itself, the experiment was repeated with the 'complementary' piece of wood, i.e. with a plank in which the rings broadened towards the other edge of the bevelling. This time the torsion was of the same sign as in the mathematical theory but the magnitude was quite out of proportion to what might be expected with uniform aeolotropy.

It is conceivable that this property of the variability of the elastic constants in wood might be made use of in those cases of practice where it is important that the flexure of a beam of asymmetrical cross-section should cause little or no torsion or a torsion in a certain direction. Although it might be difficult to select single planks that would give the suitable torsion, it might be easier to make compound torsion-free beams of several layers of different pieces of wood.

As a rough guide to the magnitude of the change due to aeolotropy, it is easy to consider the case of transverse isotropy (§ 17). If $\frac{E}{\mu}$ is fairly large so that we can neglect multiples of η in comparison with $\frac{E}{\mu}$, we see from equation (83) that the value of τ is roughly proportional to $\frac{E}{\mu}$. The value of $\frac{E}{\mu}$ for an isotropic material is $\frac{5}{2}$, but for wood it is much higher, if we take E in the direction of the wood grain. In the case of beechwood, for example, we can estimate* that $\frac{E}{\mu}$ is about 6, and for pinewood about 30.

Some measurements were made of the torsion produced in a round beam of bass-wood having a saw-cut through to the central axis. The beam had then approximately the cross-section of a complete circular sector of angle 360°. The radius of the beam was $\frac{5}{8}$ of an inch and the length of the flexed part about 36 inches. The flexing weight which was used varied up to 12 lbs.

To measure the amount of torsion one end of a stiff wire was inserted in the saw-cut at the free end of the beam so that the wire stuck out horizontally and at right

* The estimation is based on the values of the stretch-modulus given by Wertheim and Chevandier (*Comptes Rendus*, T. XXIII, p. 663, 1846) and of the Poisson's ratios found by Mallock (*Proc. Roy. Soc.*, Vol. XXIX, p. 157, 1879), and is made on the assumption of the 'partial ellipsoidal' elasticity of Saint-Venant (cf. Todhunter and Pearson's *History*, Vol. II, Part I, § [314]).

angles to the beam's axis when the beam was in its normal position. The rotation of the wire pointer was measured by noting the change in the position of its shadow cast on a piece of sectional paper by a light at some distance behind the fixed end of the beam.

Without going into the details of the measurements, it was found that the total torsion amounted to ·025 radians (or 1·5 degrees) with a flexing weight of 12 lbs. and a deflection of about $1\frac{1}{4}$ inches.

For an aeolotropic split cylinder the accurate torsional relation is

$$\tau = \frac{Wa}{EI}\left(4\frac{E}{\mu_2} + \eta_2\right) \times \cdot 10811{,}58733.$$

In the present case $\tau = \cdot 025/36$, $E = 1\cdot 470{,}000$, $a = \frac{5}{8}$ and $I = \cdot 11984{,}22489$. Thus we have $158\cdot 1896 = 4E/\mu_2 + \eta_2$. The annular rings were approximately parallel to the plane of flexure, and if we assume $\eta_2 = \cdot 45$, we have $E/\mu_2 = 39\cdot 44$, a high value, but not without parallel in the case of other woods.

For most practical purposes this method of transverse isotropy is as exact as our knowledge and the variability of the elastic constants of wood entitle us to use, but it is worth while to investigate one instance more closely with a more complete system of elastic constants, and to see the differences which arise according to the plane of aeolotropic symmetry which is taken as the plane of flexure.

Using the figures of Wertheim and Chevandier and of Mallock the following systems for beechwood and pinewood were arrived at on the assumption of the 'partial ellipsoidal' elasticity of Saint-Venant.

TABLE X. *Elastic Constants of Beechwood and Pinewood on the basis of 'partial ellipsoidal' elasticity*.*

Elastic Constants	Beechwood	Pinewood
E	980	1113
η' (perpendicular to annual rings)	·408	·372
η'' (parallel to annual rings)	·530	·486
μ'	172·9	45·3
μ''	132·7	27·1
n: the coefficient of 'partial ellipsoidal' elasticity	·754	·150

Of course no great reliance can be laid on the accuracy of these figures, but they provide systems such as are likely to occur in wood and give some idea of how in practice the aeolotropic solution given above in § 16 would work out.

The case dealt with for the isotropic solution at $\gamma = 12°$ was selected for investigation aeolotropically—that is, the 'projected' problem was taken to be that

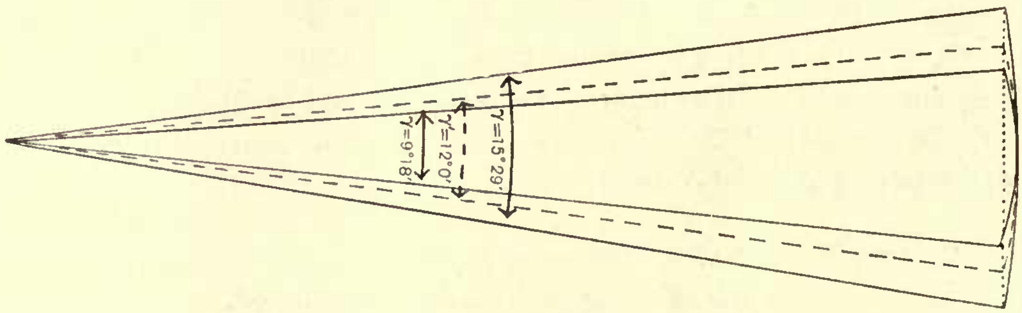
* Moduli measured in kilogrammes per sq. millimetre.

concerning a non-curtate circular sector of 12° angle. The results of the calculations are given in the following summary:

TABLE XI. *Aeolotropic case: Numerical results.*

Material	Angle	$\tau\Big/\frac{Wa}{EI}$	$\tau\Big/\frac{Wa}{EI}$ for isotropic case with the same angle as for the aeolotropic case and $\eta=\frac{1}{4}$
1°. Isotropic with $\eta=\frac{1}{4}$	12° 0′	·2846	
2°. Beechwood:			
(a) with annual rings in plane of flexure	13° 40′·9	·6828	·2837
(b) with annual rings perpendicular to plane of flexure	10° 31′·3	·5253	·2852
3°. Pinewood:			
(a) with annual rings in plane of flexure	15° 28′·6	1·9410	·2825
(b) with annual rings perpendicular to plane of flexure	9° 17′·7	1·2226	·2855

For the purposes of comparison of these aeolotropic cases with the isotropic cases of a circular sector of equal angle, it is clear that the elliptical shape of the aeolotropic sectors is not of great importance. This is illustrated in Fig. 10.



$\gamma' = 12°\ \ 0'$. Projective solution, true circular sector.
$\gamma = 15°\ 29'$. Pinewood, annual rings in plane of flexure, sector with elliptic arc.
$\gamma = \ \ 9°\ 18'$. Pinewood, annual rings perpendicular to plane of flexure, sector with elliptic arc.

FIG. 10. Diagram to illustrate closeness of elliptic sectors to true circular sectors.

The most noteworthy results of these calculations are to show (i) the effect of the direction of the annual rings in the wood on the torsion due to unbalanced shear couple, and (ii) the great increase in the flexural torsion when we pass from a material such as steel to wood. They serve to emphasise the difficulty of making applications of the mathematical theory to give numerical predictions regarding the behaviour of such a variable substance as wood. Without more precise determinations of the elastic constants of any individual plank we can at best give descriptive properties only and even then may be disappointed not to find them fully confirmed by experiment!

V. Summary of Analytical Results obtained for Beams of Cross-sections of Asymmetrical Forms other than that of the Circular Sector.

In this section are collected the analytical discussions of the problem of flexure for various other asymmetrical cross-sections, which would appear to be amenable to arithmetical, if not to complete analytical solution. The discussion is confessedly incomplete in that the results obtained are chiefly in analytical forms which can convey no immediately apparent physical meaning. To reach the stage of physical application would require months of calculation and, as long as the exigencies of more pressing war-work do not permit of the necessary leisure, we can but look forward to the arithmetical completion of the work in a continuation of the present memoir. It seems however desirable to describe here what has been done.

§ 20. *Case of a cross-section of algebraic equation derived from assuming an integral algebraic form for the function* $\chi(x, y)$. *An attempt to extend a method used by Saint-Venant.*

In looking for guidance in the works of Saint-Venant it is natural to expect that soluble cases might be found in developments of his algebraic method of solution*. The basis of the method is to start with the assumption that the function $\chi(x, y)$, which arises in the expression for the longitudinal displacement w†, is an integral algebraic function of x and y, and to find therefrom the boundary of the cross-section for which this function gives a consistent solution.

Thus, if we assume that

$$\chi(x, y) = a_0 + a_1 x + b_1 y + a_2 x^2 + 2b_2 xy + c_2 y^2 + a_3 x^3 + 3b_3 x^2 y + 3c_3 xy^2 + d_3 y^3 + \dots \qquad \text{(85)},$$

we find well-known relations between the coefficients which must hold if the equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = 0$$

is to be satisfied. When we substitute the resulting harmonic expression for χ in the boundary equation (22)

$$\widehat{xz} \cos(x\nu) + \widehat{yz} \cos(y\nu) = 0,$$

that is, in terms of the differentials dx, dy along the boundary of the cross-section

$$\widehat{xz}\, dy - \widehat{yz}\, dx = 0 \qquad \text{(86)},$$

we have the differential equation of the boundary of the cross-section.

Saint-Venant found that a large number of interesting cases, including that of

* See the *Mémoire sur la flexion* (cited on p. 4), §§ 18—23.

† Cf. equations (14) or (20) above.

the elliptic cross-section, could be discussed by retaining only the harmonic $x^3 - 3xy^2$. This leads to boundaries contained in the general equation*

$$cy^{\frac{m}{1-m}} + c_1 y^2 + x^2 = c_2 \qquad (87),$$

where m, c, c_1, c_2 are, with certain restrictions, constants at our disposal. We cannot, however, make use of this equation for any but *symmetrical* cross-sections, because in Saint-Venant's form of the shears $\widehat{yz}$ and $\widehat{xz}$ the torsion constant τ was taken to be *zero*. Thereby Saint-Venant was virtually assuming symmetry of cross-section, and he seems to have overlooked this consideration in discussing† the geometrical forms of contours which would be given by odd values of $\frac{m}{1-m}$ in equation (87). He does not however proceed to discuss in detail the flexure of a beam having any such asymmetrical cross-section.

We have therefore to follow up Saint-Venant's treatment with the more general form of equation (86) that is given in § 4 above. Omitting the details of the work, we have the result that if we take

$$\chi = -(\beta - \beta'')x - \tau xy + \frac{W}{6EI}(\eta - 2(1+\eta)m)(x^3 - 3xy^2) \qquad (88),$$

the boundary of the cross-section which corresponds is given by

$$Cy^{\frac{m}{1-m}} + \frac{W}{EI}(1+\eta)(1-m)x^2 + 2\tau y\frac{1-m}{2m-1} + \frac{W}{EI}\frac{(1-m(1+\eta))(1-m)}{3m-2}y^2 = 0 \qquad (89).$$

The latter equation may be written as

$$\zeta\left(\frac{y}{b}\right)^{\frac{m}{1-m}} + \frac{x^2}{a^2} - \xi\frac{y}{b} + \frac{y^2}{b^2} = 0 \qquad (90)$$

where m, ζ, ξ‡, a, b are constants which are not at our disposal until Conditions I and II of § 4 have been satisfied. Condition I is easily shown to be immediately satisfied, but Condition II presents greater difficulties, for the boundary as well as the integrand of the surface integrals involved depend on the constants of equation (90). As that equation stands, x is expressible in terms of a trinomial in y and it appears that in order to render the equation workable we must assume one of the coefficients to be small and so be able to expand in powers of this coefficient, the integrals of Condition II then being reducible to B-functions. This was attempted in two cases, (i) ζ assumed small, (ii) the coefficient of y^2 assumed small, but in each case the convergence of the results gave rise to doubt as to the legitimacy of the

* This integral and also that in equation (89) are obtained from equation (86) by means of the integrating factor y^n. It may be that with other integrating factors other integral harmonics might afford integrable forms, but these have yet to be found.

† At the close of § 18 of the memoir cited above.

‡ It is the absence of the linear term $\xi\frac{y}{b}$ from the equation for the boundary obtained by Saint-Venant which forms the basis of the criticism above.

assumptions by failing to give appropriate values for the quantities supposed small. Moreover the restrictions imposed on the constants of equation (90) would be much more stringent than those affecting Saint-Venant's symmetrical boundary (87) and, even if the analytical difficulties could be overcome, there would be correspondingly less interest in the solution.

§ 21. *A general method of solution by means of conjugate functions.*

The case of the circular sector cross-section with which this paper is chiefly concerned might be regarded as a solution in terms of the conjugate functions given by the relation

$$\alpha + i\beta = \log(x + iy),$$

and it is clear that the use of other sets of orthogonal curvilinear coordinates given by the general relation

$$\alpha + i\beta = f(x + iy)$$

might give rise to solutions which would come within the processes of analysis to an extent sufficient to permit of arithmetical reduction.

With a view to simplifying the analysis we may take*

$$\chi(x, y) = -\tau xy + \frac{W}{EI}(1 + \tfrac{1}{2}\eta)(xy^2 - \tfrac{1}{3}x^3) + (\alpha - \alpha'')y - (\beta_s - \beta'')x + \phi(x, y) \ldots \ldots\ldots\ldots\ldots (91)$$

where $\phi(x, y)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

With this form of χ it is seen that the boundary equation

$$\left[-\tau y + \frac{\partial\chi}{\partial x} - \frac{W}{EI}(\tfrac{1}{2}\eta x^2 + (1 - \tfrac{1}{2}\eta)y^2) + \beta_s - \beta''\right] dy$$
$$- \left[\tau x + \frac{\partial\chi}{\partial y} - \frac{W}{EI}(2 + \eta)xy - (\alpha - \alpha'')\right] dx = 0,$$

takes the simpler form

$$\left[-2\tau y - \frac{W}{EI}((1 + \eta)x^2 - \eta y^2) + \frac{\partial\phi}{\partial x}\right] dy - \frac{\partial\phi}{\partial y} dx = 0 \ldots\ldots\ldots\ldots (92).$$

Now, by the properties of conjugate functions,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\alpha}\frac{\partial\alpha}{\partial x} + \frac{\partial\phi}{\partial\beta}\frac{\partial\beta}{\partial x} = \frac{\partial\phi}{\partial\alpha}\frac{\partial\alpha}{\partial x} - \frac{\partial\phi}{\partial\beta}\frac{\partial\alpha}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial\alpha}\frac{\partial\alpha}{\partial y} + \frac{\partial\phi}{\partial\beta}\frac{\partial\beta}{\partial y} = \frac{\partial\phi}{\partial\alpha}\frac{\partial\alpha}{\partial y} + \frac{\partial\phi}{\partial\beta}\frac{\partial\alpha}{\partial x},$$

and along the contour given by $\alpha = \text{constant}$

$$\frac{\partial\alpha}{\partial x} dx + \frac{\partial\alpha}{\partial y} dy = 0.$$

* In order to avoid confusion of the slide constant β of the earlier part of this memoir with the conjugate function symbol β we now write the former β as β_s.

Hence if the contour $\alpha=$ constant is to be a part of the boundary of the cross-section of the beam, we must have, from equation (92),

$$\frac{\partial\phi}{\partial\alpha}=\frac{\dfrac{\partial\alpha}{\partial x}}{\left(\dfrac{\partial\alpha}{\partial x}\right)^2+\left(\dfrac{\partial\alpha}{\partial y}\right)^2}\left[2\tau y+\frac{W}{EI}\left((1+\eta)x^2-\eta y^2\right)\right]\text{...........}(93\,a)$$

along that part of the boundary, and similarly if $\beta=$ constant is to be a part of the boundary

$$\frac{\partial\phi}{\partial\beta}=\frac{\dfrac{\partial\beta}{\partial x}}{\left(\dfrac{\partial\beta}{\partial x}\right)^2+\left(\dfrac{\partial\beta}{\partial y}\right)^2}\left[2\tau y+\frac{W}{EI}\left((1+\eta)x^2-\eta y^2\right)\right]\text{...........}(93\,b)$$

along that part of the boundary.

The analytical problem thus becomes:

To find a function $\phi(x, y)$ such that

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}=0$$

and, therefore, equivalently by the properties of conjugate functions,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

throughout the region enclosed by a closed curve composed of parts of the contour lines given by taking α or β to be constant, and such that $\frac{\partial\phi}{\partial\alpha}$ and $\frac{\partial\phi}{\partial\beta}$ assume the values given in equations ($93\,a$) and ($93\,b$) over the parts of the boundary where $\alpha=$ constant and $\beta=$ constant respectively.

The algebraical difficulties that arise are 1°, to find the inverse expressions for x and y in terms of α and β for substitution in equations ($93a$) and ($93b$) and 2°, the determination of the Fourier solution for ϕ which will satisfy equations ($93a$) and ($93b$) in the way required. A failure to surmount these difficulties analytically would not however render the solution impossible for we can always fall back on arithmetical methods to solve any particularly interesting case. Indeed when it comes to the application of the Conditions I and II (Equations (23) and (24) of § 4), it will be necessary in all but the simplest cases to use arithmetical or mechanical methods of quadrature to determine the value of the torsion constant τ.

In the case of symmetry perpendicular to the plane of flexure Condition I is identically satisfied and Condition II becomes:

$$\tau\times\iint y^2\,dx\,dy=\frac{W}{2EI}\left[\iint\left(\eta y^3-(1+\eta)x^2y\right)dx\,dy-2(1+\eta)I\bar{y}\right]$$
$$-\tfrac{1}{2}\int\phi\left(x\cos(y\nu)-y\cos(x\nu)\right)ds\text{ ......}(94).$$

The integration with regard to ds must then be expressed as an integration with regard to α or β as the case may require.

§ 22. *Particular cases of the solution by conjugate functions.*

(i) *The Lemniscate functions*.*

These are given by

$$\alpha + i\beta = \log \frac{c}{x + i(y + c)} + \log \frac{c}{x + i(y - c)} \quad \text{.............. (95)},$$

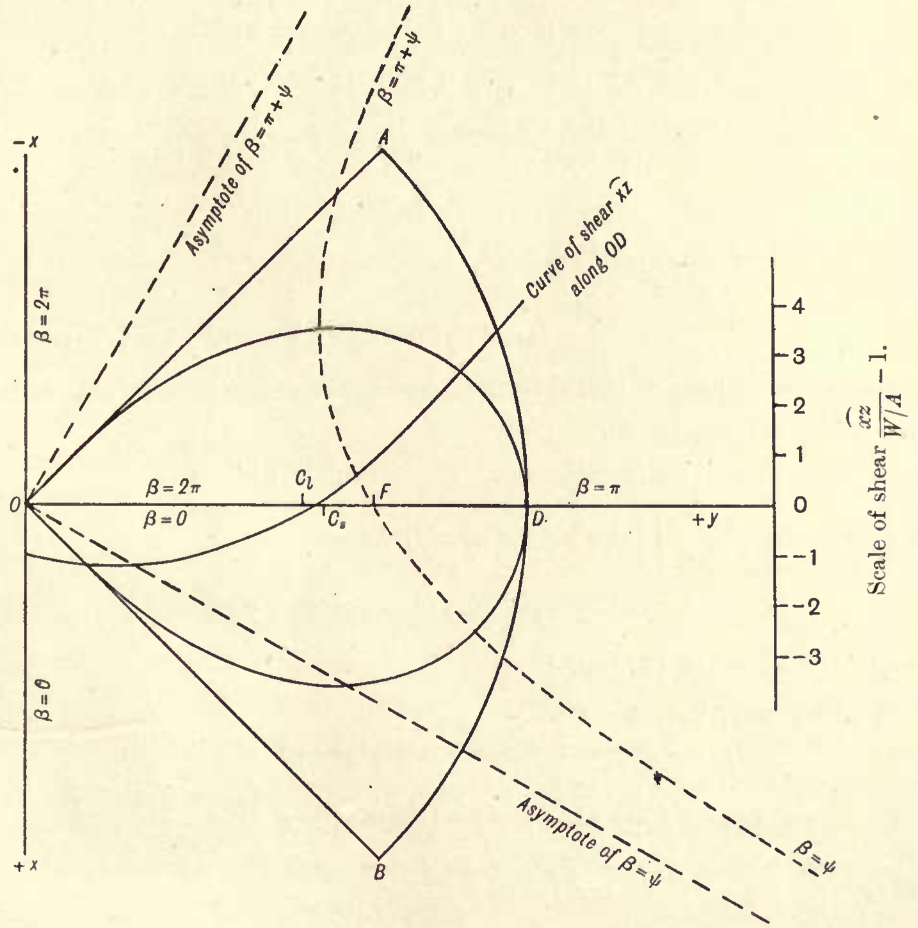
and consist of the families

$$(x^2 + y^2 - c^2)^2 + 4c^2x^2 = c^4e^{-2\alpha} \quad \text{........................ (96)}$$

and

$$x^2 - y^2 + c^2 + \frac{2xy}{\tan\beta} = 0 \quad \text{.......................... (97)},$$

the latter being a system of hyperbolas, each of which passes through the limiting points $(0, \pm c)$ or foci of the system of lemniscates given by equation (96).



C_l = centroid of lemniscate. C_s = centroid of sector. F = focus of lemniscate ($OF = c$, $OD = a = c\sqrt{2}$).

FIG. 11.

* Compare Lamé, *Leçons sur les coordonnées curvilignes*, Paris, 1859, p. 217 et seq., where will be found (p. 223) a diagram of the two orthogonal families of curves. The notation used here differs to some extent from that used by Lamé; it is arranged to provide a continuous system of β-values for the Fourier expansion. The positive half of the axis of x and origin to focus of the axis of y correspond to $\beta = 0$, the axis of y beyond the focus is $\beta = \pi$, the negative portion of the axis of x and the focus to the origin of the axis of y are both $\beta = 2\pi$. The lower branches of the hyperbolae through the focus are provided by values of β between 0 and π and the upper by values of $\beta = \pi$ to 2π, the two branches meeting at the focus correspond to ψ and $\pi + \psi$, see the diagram above.

When $\alpha=0$ equation (96) reduces to

$$(x^2+y^2)^2+2c^2(x^2-y^2)=0 \qquad (98).$$

This is the Cartesian equation of the ordinary lemniscate $r^2=2c^2\cos 2\theta$, a curve which takes the form of a figure of eight, the tangents at the double point (0, 0) being the octant lines $x=\pm y$.

We propose to find the solution of the flexure of a beam whose cross-section is *one* loop of the general lemniscate, the direction of flexure being in the direction of the axis of x and therefore at right angles to the line of symmetry of the cross-section. It is easy to show that along the boundary of this single loop we must take β to vary from 0 to 2π and thus complete a circuit of the loop.

Let
$$v=1-e^{-\alpha}\cos\beta,\quad u=\sqrt{1-2e^{-\alpha}\cos\beta+e^{-2\alpha}},$$
then
$$x^2/c^2=\tfrac{1}{2}(u-v),\quad y^2/c^2=\tfrac{1}{2}(u+v) \qquad (99).$$
and
$$\frac{d\phi}{d\alpha}=-\frac{x(x^2+y^2+c^2)}{2(x^2+y^2)}\left\{2\tau y+\frac{W}{EI}(x^2+\eta(x^2-y^2))\right\},$$
which leads to
$$\left(\frac{d\phi}{d\alpha}\right)_{\alpha=\alpha_0}=-\tfrac{1}{2}\tau c^2\sqrt{u_0^2-v_0^2}\,(1+u_0^{-1})$$
$$-\frac{Wc^3}{4EI}(u_0-v_0-2\eta v_0)\sqrt{\tfrac{1}{2}(u_0-v_0)}\,(1+u_0^{-1}) \qquad (100),$$
u_0 and v_0 being the values of u and v when $\alpha=\alpha_0$, the value of α for the required loop. This is the boundary condition.

Condition I is identically satisfied. Condition II leads to
$$2\mu\tau\iint y^2\,dx\,dy=\frac{\mu W}{EI}\iint(\eta y^3-(1+\eta)x^2y)\,dx\,dy$$
$$-W\bar{y}-\mu\int\phi_{\alpha=\alpha_0}(x\cos(y\nu)-y\cos(x\nu))\,ds \qquad (101a);$$
see (24), p. 12, and below (37), p. 17.

If $F=(x^2+y^2-c^2)^2+4c^2x^2-c^4e^{-2\alpha}$:
$$dF/dx=4x(x^2+y^2+c^2),\quad dF/dy=4y(x^2+y^2-c^2),$$
$$\cos(x\nu)=dF/dx/((dF/dx)^2+(dF/dy)^2)^{\frac{1}{2}}=\frac{x(x^2+y^2+c^2)}{\sqrt{x^2+y^2}\,c^2e^{-\alpha}},$$
and similarly
$$\cos(y\nu)=\frac{y(x^2+y^2-c^2)}{\sqrt{x^2+y^2}\,c^2e^{-\alpha}}.$$

Thus
$$\{x\cos(y\nu)-y\cos(x\nu)\}\,ds=-\frac{2xy}{\sqrt{x^2+y^2}}e^{\alpha}ds=-\frac{c\sqrt{u^2-v^2}}{\sqrt{u}}e^{\alpha}ds.$$

But
$$(ds)^2=(dx)^2+(dy)^2=\frac{c^4}{16}\left\{\left(\frac{du-dv}{x}\right)^2+\left(\frac{du+dv}{y}\right)^2\right\}$$
$$=\frac{c^2}{4}\frac{(d\beta)^2e^{-2\alpha}\sin^2\beta}{u}\frac{1+u^2-2v}{u^2-v^2}=\frac{c^2}{4}(d\beta)^2\frac{e^{-2\alpha}}{u},$$
or,
$$ds=\tfrac{1}{2}\frac{ce^{-\alpha}}{\sqrt{u}}\,d\beta.$$

Accordingly $\displaystyle\int_0^{2\pi} \phi_{\alpha=\alpha_0} (x \cos (y\nu) - y \cos (x\nu))\, ds = -\frac{c^2}{2}\int_0^{2\pi} \phi_{\alpha=\alpha_0} \frac{e^{-\alpha_0} \sin \beta\, d\beta}{u}$

$$= -\frac{c^2}{2}\int_0^{2\pi} \phi_{\alpha=\alpha_0} \frac{e^{-\alpha_0} \sin \beta}{\sqrt{1 - 2e^{-\alpha_0} \cos \beta + e^{-2\alpha_0}}}\, d\beta.$$

Thus $(101a)$ becomes

$$\tau \times 2 \iint y^2 dx dy = \frac{W}{EI}\Big[\iint (\eta y^3 - (1+\eta)\, x^2 y)\, dx dy - 2\,(1+\eta)\, I\bar{y}$$
$$+ \tfrac{1}{2} c^2 \int_0^{2\pi} \phi_{\alpha=\alpha_0} \frac{e^{-\alpha_0} \sin \beta}{\sqrt{1 - 2e^{-\alpha_0} \cos \beta + e^{-2\alpha_0}}}\, d\beta \Big] \quad\text{......(101}b\text{)}.$$

Now a solution of $$\frac{d^2\phi}{d\alpha^2} + \frac{d^2\phi}{d\beta^2} = 0$$

is $$\phi = \Sigma\, (A_n \sinh n\alpha + B_n \cosh n\alpha) \sin n (\beta - \pi) \quad\text{...........(102}a\text{)}.$$

Inside the loop $\alpha = 0$, α ranges from 0 to $+\infty$ at the focus $(0, c)$, but w and therefore ϕ must be finite; it follows therefore that $A_n + B_n$ must be zero and our solution will take the form

$$\phi = \Sigma\, (A_n e^{-n\alpha} \sin n\, (\beta - \pi)) \quad\text{......................(102}b\text{)},$$

$$\left(\frac{d\phi}{d\alpha}\right)_{\alpha=\alpha_0} = -\Sigma\, (n A_n e^{-n\alpha_0} \sin n\, (\beta - \pi)) \quad\text{..................(103)}.$$

We have accordingly, from (100) and (103),

$$\Sigma\, (n A_n e^{-n\alpha_0} \sin n (\beta - \pi)) = \tfrac{1}{2} \tau c^2 f(\beta) + \frac{Wc^2}{4EI} \psi(\beta),$$

where $\quad f(\beta) = \sqrt{u_0^2 - v_0^2}\,(1 + u_0^{-1}), \quad \psi(\beta) = (u_0 - v_0 - 2\eta v_0) \sqrt{\tfrac{1}{2}(u_0 - v_0)}\,(1 + u_0^{-1}).$

If $$A_n = \tfrac{1}{2} \tau c^2 A_n' + \tfrac{1}{4} \frac{Wc^3}{EI} A_n'',$$

it follows that $$A_n' = \frac{e^{n\alpha_0}}{n\pi + \sin n\pi} \int_0^{2\pi} f(\beta) \sin n\, (\beta - \pi)\, d\beta,$$

$$A_n'' = \frac{e^{n\alpha_0}}{n\pi + \sin n\pi} \int_0^{2\pi} \psi(\beta) \sin n\, (\beta - \pi)\, d\beta.$$

These integrals must be obtained by quadratures and the values of A_n' and A_n'' then inserted in the β-integral term of the torsion equation $(101b)$, where again for the general case quadrature is needful. The proper values for n can only be determined when we have found the expressions $f(\beta)$ and $\psi(\beta)$ for the particular case.

We will illustrate the method on the relatively simple problem of the torsion flexure of a beam, the section of which is the loop of an ordinary lemniscate. In this case $\alpha_0 = 0$ and

$$\int_0^{2\pi} \phi_{\alpha=\alpha_0} \frac{e^{-\alpha_0} \sin \beta}{\sqrt{1 - 2e^{-\alpha_0} \cos \beta + e^{-2\alpha_0}}}\, d\beta = \int_0^{2\pi} \phi_{\alpha=0} \cos \tfrac{1}{2}\beta\, d\beta.$$

Now in this case

$$v_0 = 2 \sin^2 \tfrac{1}{2}\beta, \quad u_0 = 2 \sin \tfrac{1}{2}\beta,$$

and we find

$$x = c \sin \tfrac{1}{4} (\pi - \beta) \sqrt{2 \sin \tfrac{1}{2}\beta}, \quad y = c \cos \tfrac{1}{4} (\pi - \beta) \sqrt{2 \sin \tfrac{1}{2}\beta} \quad\text{......(104)}$$

over the contour $\alpha = 0$, and accordingly

$$\left(\frac{d\phi}{d\alpha}\right)_{\alpha=0} = -\tfrac{1}{2}\tau c^2 (\sin\beta + \cos\tfrac{1}{2}\beta)$$

$$-\frac{Wc^3}{4EI}\sqrt{2\sin\tfrac{1}{2}\beta}\{\cos\tfrac{1}{4}(3\beta-\pi) + (\tfrac{1}{2}+\eta)(\sin\tfrac{1}{4}(\beta-\pi) - \sin\tfrac{1}{4}(5\beta-\pi))\} \quad\text{......(104}b\text{)}.$$

Thus $\quad f(\beta) = \sin\beta + \cos\tfrac{1}{2}\beta,$

$$\psi(\beta) = \sqrt{2\sin\tfrac{1}{2}\beta}\{\cos\tfrac{1}{4}(3\beta-\pi) + (\tfrac{1}{2}+\eta)(\sin\tfrac{1}{4}(\beta-\pi) - \sin\tfrac{1}{4}(5\beta-\pi))\}.$$

A suitable form of ϕ is

$$\phi = \tfrac{1}{2}\tau c^2 (e^{-\alpha}\sin\beta + 2e^{-\frac{1}{2}\alpha}\cos\tfrac{1}{2}\beta)$$

$$+\sum_1^\infty \left\{(A_n''' + (\tfrac{1}{2}+\eta)A_n^{\text{iv}})\frac{Wc^3}{4EI}\times e^{-n\alpha}\sin n(\beta-\pi)\right\} \quad\text{......(105)}.$$

This will satisfy (104b) provided we take

$$\left.\begin{aligned} A_n''' &= \frac{1}{n\pi}\int_0^{2\pi}\sqrt{2\sin\tfrac{1}{2}\beta}\cos\tfrac{1}{4}(3\beta-\pi)\sin n(\beta-\pi)\,d\beta \\ A_n^{\text{iv}} &= \frac{1}{n\pi}\int_0^{2\pi}\sqrt{2\sin\tfrac{1}{2}\beta}(\sin\tfrac{1}{4}(\beta-\pi) - \sin\tfrac{1}{4}(5\beta-\pi))\sin n(\beta-\pi)\,d\beta \end{aligned}\right\} \text{...(106)}.$$

These may be shown to be

$$\left.\begin{aligned} A_n''' &= -\frac{2}{n\pi}\int_0^{\pi}\sqrt{2\cos\tfrac{1}{2}\beta'}\sin\tfrac{3}{4}\beta'\sin n\beta'\,d\beta' \\ A_n^{\text{iv}} &= \frac{2}{n\pi}\int_0^{\pi}(2\cos\tfrac{1}{2}\beta')^{\frac{3}{2}}\sin\tfrac{3}{4}\beta'\sin n\beta'\,d\beta' \end{aligned}\right\} \text{............(107)}.$$

These integrals were found by quadratures, using one of Sheppard's formulae (*Biometrika*, Vol. I, p. 276, Case (i) (c)), the subject of integration being tabled to every 5°. After considerable labour there resulted :

n	A_n'''	A_n^{iv}	$\frac{n(-1)^n}{4n^2-1}$	$\kappa_n = \frac{1}{4}A_n''' + \frac{1}{8}A_n^{\text{iv}}$	$\kappa_n' = \frac{1}{4}A_n^{\text{iv}}$
1	− 1·125,0142	+ 1·500,0012	− ·333,3333	− ·093,7535	+ ·375,0003
2	+ ·031,2632	+ ·187,4978	+ ·133,3333	+ ·031,2530	+ ·046,8745
3	− ·013,0284	− ·020,8406	− ·085,7143	− ·005,8622	− ·005,2102
4	+ ·006,8490	+ ·005,8517	+ ·063,4921	+ ·002,4437	+ ·001,4629
5	− ·004,0996	− ·002,3637	− ·050,5051	− ·001,3204	− ·000,5909
6	+ ·002,6992	+ ·001,1255	+ ·041,9580	+ ·000,8155	+ ·000,2814
7	− ·001,8626	− ·000,6602	− ·035,8974	− ·000,5482	− ·000,1650

Now $\quad \phi_{\alpha=0} = \tfrac{1}{2}\tau c^2(\sin\beta + 2\cos\tfrac{1}{2}\beta) + \sum_1^\infty (A_n''' + (\tfrac{1}{2}+\eta)A_n^{\text{iv}})\frac{Wc^3}{4EI}\sin n(\beta-\pi)$

and $\quad \int_0^{2\pi}\phi_{\alpha=0}\cos\tfrac{1}{2}\beta\,d\beta = \tfrac{1}{2}\tau c^2(\tfrac{8}{3}+2\pi) + \frac{Wc^3}{4EI}\sum_1^\infty\left\{(A_n''' + (\tfrac{1}{2}+\eta)A_n^{\text{iv}})\frac{2n(-1)^n}{n^2-\frac{1}{4}}\right\}.$

Equation (101b) for the torsion now becomes

$$\tau\left\{2\iint y^2 dx\,dy - \tfrac{1}{4}c^4\left(\tfrac{8}{3}+2\pi\right)\right\}$$
$$=\frac{W}{EI}\left[\iint\{\eta y^3-(1+\eta)x^2y\}\,dx\,dy-2(1+\eta)I\bar{y}\right.$$
$$\left.+c^5\sum_1^\infty\left\{\left(A_n'''+(\tfrac{1}{2}+\eta)A_n^{\text{iv}}\right)\frac{n(-1)^n}{4n^2-1}\right\}\right]\quad\text{......(108).}$$

But for the common lemniscate

$$\iint y^2dx\,dy=\tfrac{1}{2}c^4\left(\tfrac{1}{4}\pi+\tfrac{2}{3}\right),\quad \iint y^3dx\,dy=\tfrac{15}{64}\pi c^5,\quad \iint x^2y\,dx\,dy=\tfrac{1}{64}\pi c^5,$$
$$I=\tfrac{1}{2}c^4\left(\tfrac{1}{4}\pi-\tfrac{2}{3}\right),\quad \bar{y}=\tfrac{1}{4}\pi c,$$

thus we have

$$\tau\left(-\tfrac{1}{4}\pi\right)=\frac{Wc}{EI}\left\{\eta\frac{\pi}{4}\left(\frac{74}{48}-\frac{\pi}{4}\right)-\frac{\pi}{4}\left(\frac{\pi}{4}-\frac{29}{48}\right)+\cdot381{,}1119-\left(\tfrac{1}{2}+\eta\right)\cdot472{,}6525\right\},$$

$$\tau\left(-\cdot7853{,}9816\right)=\frac{Wc}{EI}\{\cdot002{,}4468+\cdot121{,}3194\eta\},$$

$$\tau=-\frac{Wc}{EI}\{\cdot003{,}1154+\cdot154{,}4687\eta\}\quad\text{......(109)}$$

$$=-\frac{Wa}{EI}\{\cdot002{,}203+\cdot109{,}2259\eta\},\ \text{if } a=\sqrt{2}c,$$

the breadth of the lemniscate. If $\eta=\frac{1}{4}$,

$$\tau=-\frac{Wa}{EI}\times\cdot02951.$$

Thus if the lengthy analysis be without slip, there is a small *negative* torsion, the first we have come across.

We can now complete the solution. We will write

$$\tau=-\frac{Wc}{EI}(\epsilon_1+\epsilon_2\eta),$$

and we have

$$\phi(x,y)=-\frac{Wc^3}{EI}\left[\left(\tfrac{1}{2}e^{-\alpha}\sin\beta+e^{-\frac{1}{2}\alpha}\cos\tfrac{1}{2}\beta\right)(\epsilon_1+\epsilon_2\eta)\right.$$
$$\left.-\sum_1^\infty\left\{\left(\tfrac{1}{4}A_n'''+\tfrac{1}{8}A_n''+\tfrac{1}{4}\eta A_n^{\text{iv}}\right)e^{-n\alpha}\sin n(\beta-\pi)\right\}\right]$$
$$=-\frac{Wc^3}{EI}\left[\left(\tfrac{1}{2}e^{-\alpha}\sin\beta+e^{-\frac{1}{2}\alpha}\cos\tfrac{1}{2}\beta\right)(\epsilon_1+\epsilon_2\eta)-\sum_1^\infty\{(\kappa_n+\kappa_n'\eta)e^{-n\alpha}\sin n(\beta-\pi)\}\right],$$

the numerical values of κ_n, κ_n' being given on p. 64. Hence by equations (20) and (91)

$$w=\frac{Wc^3}{EI}\left[(\epsilon_1+\epsilon_2\eta)\left(\frac{xy}{c^2}-\tfrac{1}{2}e^{-\alpha}\sin\beta-e^{-\frac{1}{2}\alpha}\cos\tfrac{1}{2}\beta\right)+\frac{\frac{1}{2}\eta xy^2-\frac{1}{3}\left(1+\frac{1}{2}\eta\right)x^3}{c^3}\right.$$
$$\left.-\frac{x\left(lz-\frac{1}{2}z^2\right)}{c^3}+\sum_1^\infty(\kappa_n+\kappa_n'\eta)e^{-n\alpha}\sin n(\beta-\pi)\right]-\beta_s x+\alpha_s y+\gamma',$$

u and v will be as in equation (20), p. 12.

We have now to determine the fixing of the clamped end; let us suppose the element at the centroid fixed. Then, by p. 42, $\alpha_s = \beta' = 0$ and

$$\gamma = -\frac{W}{EI}\eta l\bar{y}, \quad \gamma' = (\epsilon_1 + \epsilon_2 \eta),$$

$$-\frac{Wl}{EI}\tfrac{1}{2}\eta\bar{y}^2 - \gamma\bar{y} + \alpha' = 0.$$

Hence
$$u(0, \bar{y}) = -\tau\bar{y}z + \frac{W}{EI}[\tfrac{1}{2}\eta\bar{y}^2 z + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3] + \beta_s z$$

$$= \frac{W}{EI}(\tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3) + z\left(\beta_s - \tau\bar{y} + \tfrac{1}{2}\eta\bar{y}^2\frac{W}{EI}\right),$$

or
$$s_0 = \beta_s - \tau\bar{y} + \tfrac{1}{2}\eta\bar{y}^2\frac{W}{EI}$$

gives the linear term in the deflection as soon as β_s is found.

We have now to find $\frac{dw}{dx}$ along the axis of symmetry. Clearly $\frac{d\alpha}{dx} = 0$, when $x = 0$,

$$\frac{d\beta}{dx} = -\frac{2y\cos^2\beta}{c^2 - y^2}, \text{ when } x = 0.$$

But $e^{-\alpha} = (c^2 - y^2)/c^2$ for $y < c$, $= (y^2 - c^2)/c^2$ for $y > c$, and at the centroid the former must be used for $\bar{y} = \frac{1}{4}\pi c$. Further $\beta = 0$ for $x = 0$, from the origin to the focus, and $= \pi$ beyond the focus, i.e. $y = c$. Thus between origin and focus

$$\frac{dw}{dx} = \frac{Wc^2}{EI}\left[(\epsilon_1 + \epsilon_2\eta)\left(\frac{y}{c} - \frac{y}{c}\right) + \tfrac{1}{2}\eta\frac{y^2}{c^2} - \frac{lz - \frac{1}{2}z^2}{c^2}\right.$$
$$\left. - \frac{y}{c}\sum_1^\infty 2n(-1)^n e^{-(n-1)\alpha}(\kappa_n + \kappa_n'\eta)\right] - \beta_s.$$

Further, by equation (20) along $x = 0$,

$$\frac{du}{dz} = -\tau y + \frac{W}{EI}[\tfrac{1}{2}\eta y^2 + lz - \tfrac{1}{2}z^2] + \beta_s.$$

Hence adding

$$\left(\frac{\widehat{xz}}{\mu}\right)_{\substack{x=0\\y<c}} = \frac{dw}{dx} + \frac{du}{dz} = \frac{Wc^2}{EI}\left[(\epsilon_1 + \epsilon_2\eta)\frac{y}{c} + \eta\frac{y^2}{c^2}\right.$$
$$\left. - \frac{y}{c}\sum_1^\infty 2n(-1)^n e^{-(n-1)\alpha}(\kappa_n + \kappa_n'\eta)\right] \text{.........(110)},$$

which enables us to get the shearing stress from the origin to the focus. Beyond the focus $\beta = \pi$, thus $\cos\beta$ changes sign, and $c^2 - y^2 = -c^2 e^{-\alpha}$, thus the term $\frac{1}{2}e^{-\alpha}\sin\beta$ in w remains the same as before in dw/dx, but the last term changes and we have

$$\left(\frac{\widehat{xz}}{\mu}\right)_{\substack{x=0\\y>c}} = \frac{Wc^2}{EI}\left[(\epsilon_1 + \epsilon_2\eta)\frac{y}{c} + \eta\frac{y^2}{c^2} + \frac{y}{c}\sum_1^\infty 2ne^{-(n-1)\alpha}(\kappa_n + \kappa_n'\eta)\right] \text{......}(110)^{bis}.$$

At the focus $\alpha=\infty$ and the summation reduces to a single term the same for both; thus we have

$$\left(\frac{\widehat{xz}}{\mu}\right)_{\substack{z=0\\ y=c}} = \frac{Wc^2}{EI}\left[\epsilon_1+2\kappa_1+(\epsilon_2+1+2\kappa_1')\,\eta\right]$$

$$= \frac{Wc^2}{EI}\left[-\cdot 184{,}3916+1\cdot 904{,}4693\eta\right] \ldots\ldots\ldots\ldots\ldots (111)$$

$$= \frac{Wc^2}{EI}\times\cdot 29173,\ \text{if}\ \eta=\tfrac{1}{4},\ = \frac{Wa^2}{EI}\times\cdot 14586,\ \text{if}\ a=\sqrt{2}c.$$

At the centroid we have

$$\beta_s=\frac{Wc^2}{EI}\left[\tfrac{1}{2}\eta\,\frac{\pi^2}{16}-\sum_1^\infty(\kappa_n+\kappa_n'\eta)\,\frac{\pi n e^{-(n-1)\alpha}(-1)^n}{2}\right],$$

and

$$e^{-\alpha}=(c^2-\bar{y}^2)/c^2=1-\tfrac{1}{16}\pi^2=\cdot 3831{,}4973.$$

But

$$\left(\frac{\widehat{xz}}{\mu}\right)_{\substack{x=0\\ y=\bar{y}}} = \frac{Wc^2}{EI}\left[(\epsilon_1+\epsilon_2\eta)\,\tfrac{1}{4}\pi+\eta\,\frac{\pi^2}{16}-\sum_1^\infty(\kappa_n+\kappa_n'\eta)\,\frac{\pi n e^{-(n-1)\alpha}(-1)^n}{2}\right]$$

$$=\tfrac{1}{2}\eta\,\frac{W\bar{y}^2}{EI}-\tau\bar{y}+\beta_s=s_0,$$

or, the curve of deflection is

$$u\,(0,\bar{y})=\frac{W}{EI}\left(\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right)+z\left(\frac{\widehat{xz}}{\mu}\right)_{\substack{x=0\\ y=\bar{y}}} \ldots\ldots\ldots\ldots\ldots\ldots (112),$$

as it should be. It only remains to find β_s and s_0.

Numerically we have

n	$\frac{1}{2}\pi n e^{-(n-1)\alpha}(-1)^n$	$\kappa_n\times\frac{1}{2}\pi n e^{-(n-1)\alpha}(-1)^n$	$\kappa_n'\times\frac{1}{2}\pi n e^{-(n-1)\alpha}(-1)^n$
1	− 1·570,7963	+ ·147,2677	− ·589,0491
2	+ 1·203,7003	+ ·037,6192	+ ·056,4228
3	− ·691,7963	+ ·004,0554	+ ·003,6044
4	+ ·353,4154	+ ·000,8636	+ ·000,5170
5	− ·169,2637	+ ·000,2235	+ ·000,1000
6	+ ·077,8238	+ ·000,0635	+ ·000,0219
7	− ·036,3589	+ ·000,0199	+ ·000,0060

These values lead us to

$$\beta_s=\frac{Wc^2}{EI}\left[\cdot 308{,}4251\eta-\cdot 190{,}1128+\cdot 528{,}3770\eta\right]$$

$$=\frac{Wc^2}{EI}\left[-\cdot 190{,}1128+\cdot 836{,}8021\eta\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (113)$$

$$=\frac{Wa^2}{EI}\times\cdot 009{,}544,$$

if $\eta=\frac{1}{4}$ and $a=\sqrt{2}c$ be the breadth of the lemniscate loop.

Further,

$$s_0=\frac{Wc^2}{EI}\left[1\cdot 266{,}5467\eta-\cdot 187{,}6660\right] \ldots\ldots\ldots\ldots\ldots\ldots (114)$$

$$=\frac{Wa^2}{EI}\times\cdot 06449,$$

if $\eta=\frac{1}{4}$ and $a=\sqrt{2}c$ be the breadth of the lemniscate.

Thus the deflection will be

$$\delta_z = \frac{Wl^3}{3EI}\left[\tfrac{3}{2}\left(\frac{z}{l}\right)^2 - \tfrac{1}{2}\left(\frac{z}{l}\right)^3 + 3\frac{a^2}{l^2}\cdot 06449\left(\frac{z}{l}\right)\right] \quad\text{...............(115)}.$$

The linear term is only about $\frac{1}{5}$th of the linear term in a circular sector of angle 90°. This is due to the change in the torsion from a sensible positive to a slight negative value.

It remains to investigate the vertical shear along the central radius. In working this out attention must be paid to the change in form in the shear after passing the focus. The following table gives the chief results:

Shears parallel to load along Axis of Symmetry.

y/c	y/a	$\widehat{xz} \Big/ \frac{Wa^2}{EI}$	$\eta = \frac{1}{4}$	
			$\widehat{xz} \Big/ \frac{Wa^2}{EI}$	$\widehat{xz}$/(mean shear) − 1
0	0	0	0	− 1
·1	·0707	− ·019,541 + ·038,289η	− ·0100	− 1·135
·2	·1414	− ·037,932 + ·087,573η	− ·0160	− 1·216
·4	·2828	− ·067,834 + ·222,562η	− ·0122	− 1·164
·6	·4243	− ·086,317 + ·410,126η	+ ·0162	− ·782
$\frac{1}{4}\pi$	·5554	− ·093,833 + ·633,273η	+ ·0645	− ·131
1	·7071	− ·092,196 + ·952,235η	+ ·1459	+ ·966
1·15	·8132	− ·084,637 + 1·214,414η	+ ·2190	+ 1·951
1·30	·9192	− ·071,986 + 1·509,133η	+ ·3053	+ 3·114
$\sqrt{2}$	1·0	− ·060,879 + 1·754,884η	+ ·3778	+ 4·091

The last column is plotted on the diagram, p. 61. It is clear that from the origin onwards we have a small negative shear and ultimately a moderately large positive shear, reaching a maximum at the butt end of the section. The form of the distribution differs widely from that of the 90° circular sector. We have not succeeded in discovering any physical reason for the torsion being negative in the case of the lemniscate and positive in the case of the circular sector. On the other hand, we have not discovered after much investigation any slip in the analysis. We publish the investigation in the hope that a fresher eye may discover our slip or verify the analysis, or again find other sections with negative torsion. Should such sections actually exist, the quick transition in sign with changes of not very marked character in the asymmetry might suggest that the edge of attack in a propeller-blade may be of a rather unstable nature. Possibly when time is available for the sections suggested below to be worked out numerically, light will be thrown on this interesting question.

(ii) *The coaxial circle functions**.

The equation

$$\alpha + i\beta = \log\frac{x + i(y + c)}{x + i(y - c)} \quad\text{........................(116)}$$

* 'Système cylindrique bicirculaire' of Lamé: see his *Leçons sur les coordonnées curvilignes*, p. 199, Paris, 1859.

gives rise to the two families of circles

$$x^2+(y-c\coth\alpha)^2=c^2\operatorname{cosech}^2\alpha \quad\text{.....................}(117),$$

$$(x-c\cot\beta)^2+y^2=c^2\operatorname{cosec}^2\beta \quad\text{.....................}(118),$$

the α-family being a system of coaxial circles having real limiting points at $(0, \pm c)$ and the axis of x as radical axis, and the β-family being the orthogonal system of coaxial circles.

The interesting case from the present point of view is the boundary given by taking two circles of the α-family, one being wholly inside the other*. The beam then is a hollow shaft with an eccentric cavity, and thus possesses the requisite uni-axial symmetry.

The expressions we require for substitution are:

$$\left.\begin{aligned} x&=\frac{c\sin\beta}{\cosh\alpha-\cos\beta}\\ y&=\frac{c\sinh\alpha}{\cosh\alpha-\cos\beta}\end{aligned}\right\} \quad\text{..................}(119),$$

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial x}&=-\frac{\sinh\alpha\sin\beta}{c}\\ \frac{\partial\alpha}{\partial y}&=\frac{1-\cosh\alpha\cos\beta}{c}\\ \left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2&=\frac{(\cosh\alpha-\cos\beta)^2}{c^2}\end{aligned}\right\} \quad\text{..................}(120),$$

and it is easily seen that a complete circuit of any circle of the α-family is given by taking β to vary from $-\pi$ to $+\pi$.

Equation (93) becomes for this case

$$\frac{\partial\phi}{\partial\alpha}=-2\tau c^2\frac{\sinh^2\alpha\sin\beta}{(\cosh\alpha-\cos\beta)^2}-\frac{Wc^3}{EI}\frac{(1+\eta)\sinh\alpha\sin^3\beta-\eta\sinh^3\alpha\sin\beta}{(\cosh\alpha-\cos\beta)^3} \quad\text{..................}(121),$$

and the form of the solution will be†

$$\phi=\sum_{n=1}^{\infty}\frac{A_n\cosh n(\alpha_1-\alpha)+B_n\cosh n(\alpha_2-\alpha)}{\sinh n(\alpha_1-\alpha_2)}\sin n\beta \quad\text{.........}(122),$$

where the constants A_n and B_n are determined by Fourier's method from the equations obtained by substituting α_1 and α_2 for α in equation (121).

Condition I is again identically satisfied. Condition II becomes

$$\tau\int y^2dx\,dy=\frac{W}{2EI}\left[\iint(\eta y^3-(1+\eta)x^2y)\,dx\,dy-2(1+\eta)I\bar{y}\right]$$
$$+\tfrac{1}{2}c^2\int_{-\pi}^{+\pi}\frac{\sin\beta\cosh\alpha}{\cosh\alpha-\cos\beta}\phi\,d\beta \quad\text{......}(123).$$

* This problem is suggested as a soluble one by Love (*Elasticity*, Second Edition, § 232 (*e*), p. 325). The corresponding torsion problem has been investigated by H. M. Macdonald, *Proc. Camb. Phil. Soc.*, Vol. VIII., p. 62, 1893.

† Lamé, *loc. cit.*, p. 184.

In the β-integral α in ϕ and $\cosh \alpha$ must be given the two selected values and the integrations round the two circles taken in opposite senses. As in the case of the lemniscate ϕ will contain series terms with both τ and W/EI as coefficients.

To get the solution for a beam resembling in cross-section the blade of a propeller we might use the curvilinear quadrilateral enclosed between the circular arcs given by α_1, α_2, β_1, $-\beta_1$, but in the sector of an annulus we have such a similar figure that there would be little but the pleasure of confirmation in following out this case.

(iii) *The confocal conic functions**.

The appropriate systems for our problem are given by

$$\alpha + i\beta = \sinh^{-1}\frac{1}{c}(x+iy) \quad \text{.......................(124)},$$

and consist of the family of ellipses with foci at $(0, \pm c)$

$$\frac{x^2}{\sinh^2\alpha} + \frac{y^2}{\cosh^2\alpha} = c^2 \quad \text{..........................(125)},$$

and the family of hyperbolas with the same foci

$$-\frac{x^2}{\cos^2\beta} + \frac{y^2}{\sin^2\beta} = c^2 \quad \text{..........................(126)}.$$

The Cartesian coordinates of the system are

$$\left.\begin{aligned} x &= c\sinh\alpha\cos\beta \\ y &= c\cosh\alpha\sin\beta \end{aligned}\right\} \quad \text{..........................(127)},$$

and it is seen that the part of the axis of y lying between the foci may be taken as the ellipse $\alpha = 0$, and the part lying outside of the foci as the hyperbola $\beta = \frac{1}{2}\pi$, and the positive axis of x as the hyperbola $\beta = 0$ and the negative axis of x as the hyperbola $\beta = \pi$.

From the point of view of the flexure problem, and especially in relation to the problem of the propeller-blade, it would appear that the most interesting case would be the boundary enclosed between an ellipse α_0 and one limb of a hyperbola given by taking $\beta = \frac{\pi}{2} - \beta_0$, where β_0 has an appropriately small value so that the eccentricity of the hyperbola will be fairly large. As in all cases, however, where composite boundaries are made up from systems of orthotomic curves, there is the disadvantage of the right-angled corners which do not appear in the section of a propeller-blade.

For this case, where $-\alpha_0 < \alpha < \alpha_0$ and $\frac{\pi}{2} - \beta_0 < \beta < \frac{\pi}{2} + \beta_0$, the boundary conditions derived from equation (93) are easily found to be

$$\frac{\partial\phi}{\partial\alpha} = 2\tau c^2\cosh^2\alpha\sin\beta\cos\beta + \frac{Wc^3}{EI}\left((1+\eta)\sinh^2\alpha\cosh\alpha\cos^3\beta - \eta\cosh^3\alpha\sin^2\beta\cos\beta\right)$$
$$\text{........................(128)}$$

* 'Cylindres homofocaux' of Lamé, *loc. cit.*, p. 195.

and

$$\frac{\partial\phi}{\partial\beta} = -2\tau c^2 \sinh\alpha\cosh\alpha\sin^2\beta - \frac{Wc^3}{EI}((1+\eta)\sinh^3\alpha\sin\beta\cos^2\beta - \eta\sinh\alpha\cosh^2\alpha\sin^3\beta) \qquad \text{.........................(129)},$$

and the solution for ϕ will be of the form

$$\phi = A_n \sinh\frac{n\pi}{\alpha_0}\left(\beta - \frac{\pi}{2} + \beta_0\right)\cos\frac{n\pi}{\alpha_0}(\alpha-\alpha_0) + B_n\sinh\frac{n\pi}{\beta_0}(\alpha-\alpha_0)\cos\frac{n\pi}{\beta_0}\left(\beta-\frac{\pi}{2}+\beta_0\right) \qquad \text{.........................(130)},$$

where the constants A_n are derived by Fourier's method from equation (129) for $\beta = \frac{\pi}{2} \pm \beta_0$ and the constants B_n from equation (128) for $\alpha = \alpha_0$.

Turning to (94) the $\int \phi\,(x\cos(y\nu) - y\cos(x\nu))\,ds = -\frac{1}{2}c^2\int \sin 2\beta\,\phi_{\alpha_0}\,d\beta$ along an $\alpha = \alpha_0$ boundary, and to $+\frac{1}{2}c^2\int\sinh 2\alpha\,\phi_{\beta_1}\,d\alpha$ along a β_1 boundary. Thus the equation for the torsion is provided as soon as A_n and B_n have been determined.

It is evident that this solution can be completely set forth ready for arithmetical calculation without methods of mechanical integration.

(iv) *The rosette functions.*

We shall lastly consider the system of conjugate families given by

$$\alpha + i\beta = \frac{(ai)^m}{(x+iy)^m} \qquad \text{.....................................(131)}.$$

When polar coordinates $(x = r\sin\theta,\ y = r\cos\theta)$* are used the equations of the two families are found to be

$$\left.\begin{aligned} \alpha &= \frac{a^m}{r^m}\cos m\theta \\ \beta &= \frac{a^m}{r^m}\sin m\theta \end{aligned}\right\} \qquad \text{...............................(132)},$$

and we easily deduce that

$$\left.\begin{aligned} x &= a(\alpha^2+\beta^2)^{-\frac{1}{2m}}\sin\left(\frac{1}{m}\tan^{-1}\frac{\beta}{\alpha}\right) \\ y &= a(\alpha^2+\beta^2)^{-\frac{1}{2m}}\cos\left(\frac{1}{m}\tan^{-1}\frac{\beta}{\alpha}\right) \end{aligned}\right\} \qquad \text{.....................(133)},$$

and that

$$\begin{aligned} \frac{\dfrac{\partial\alpha}{\partial x}}{\left(\dfrac{\partial\alpha}{\partial x}\right)^2 + \left(\dfrac{\partial\alpha}{\partial y}\right)^2} &= -\frac{r^{m+1}}{ma^m}\sin(m+1)\theta \\ &= -\frac{a}{m}(\alpha^2+\beta^2)^{-1-\frac{1}{2m}}\left(\beta\cos\left(\frac{1}{m}\tan^{-1}\frac{\beta}{\alpha}\right) + \alpha\sin\left(\frac{1}{m}\tan^{-1}\frac{\beta}{\alpha}\right)\right) \quad \text{...(134)}. \end{aligned}$$

* An unusual form of polar coordinates has been adopted in order to preserve uniformity with the notation given in the earlier portion of the paper.

The typical curve of the α-family is a rosette composed of petals radiating from the origin and enclosed within a circle of radius $a\alpha^{-\frac{1}{m}}$, which is touched by each petal. Let us take the rosette $\alpha=1$ and let us consider the single petal which results from the principal values of $\tan^{-1}\beta$ being used, that is for $-\frac{\pi}{2}\leqslant\tan^{-1}\beta\leqslant+\frac{\pi}{2}$. This petal is a closed curve symmetrical about the axis of y and having a node of angle $\frac{\pi}{m}$ at the origin. With a suitable choice of m we thus have a curve which would resemble fairly closely the cross-section of a propeller-blade, and it is with this in view that we proceed to give a formal solution of the flexure problem of a beam having such a cross-section.

The boundary equation (93) becomes after reduction

$$\left(\frac{\partial\phi}{\partial\alpha}\right)_{\alpha=1}=-\left[\frac{2\tau a^2}{m}(1+\beta^2)^{-1-\frac{1}{m}}\cos\left(\frac{1}{m}\tan^{-1}\beta\right)+\frac{Wa^3(1+\beta^2)^{-1-\frac{3}{2m}}}{EI\quad m}\right.$$
$$\left.\times\left((1+\eta)\sin^2\left(\frac{1}{m}\tan^{-1}\beta\right)-\eta\cos^2\left(\frac{1}{m}\tan^{-1}\beta\right)\right)\right]\left(\beta\cos\left(\frac{1}{m}\tan^{-1}\beta\right)+\sin\left(\frac{1}{m}\tan^{-1}\beta\right)\right)\quad\text{.........................(135)},$$

and this is to be satisfied for the whole of the range $-\infty<\beta<\infty$, a range which renders a Fourier *series* solution inadequate.

Consider the integral

$$\phi(\alpha,\beta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}F(v)\left(\frac{e^{-u}}{u}-\frac{e^{-u(\alpha-1)}}{u}\cos u(v-\beta)\right)du\,dv\quad\text{...(136)}.$$

The integrand is everywhere finite if $F(v)$ is finite for $-\infty<v<\infty$, and it will be assumed for the present that the integral is convergent* for the range $0\leqslant\alpha\leqslant1$, $-\infty<\beta<\infty$. Also it satisfies the differential equation

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}=0,$$

and we have

$$\left(\frac{\partial\phi}{\partial\alpha}\right)_{\alpha=1}=\frac{1}{\pi}\int_{-\infty}^{\infty}F(v)\int_0^{\infty}\cos u(v-\beta)\,du\,dv\quad\text{...............(137)}$$
$$=F(\beta),\text{ by Fourier's integral theorem.}$$

If, therefore, for $F(v)$ in equation (136) we substitute the expression for

$$\left(\frac{\partial\phi(\alpha,v)}{\partial\alpha}\right)_{\alpha=1}$$

* It does not appear to have been noticed by Fourier that the integral $\int\frac{e^{-u(\alpha-1)}}{u}\cos u(v-\beta)\,du$ is divergent for any range of integration which includes the point $u=0$. Compare the *Théorie analytique de la Chaleur*, § 421, Darboux's edition of the *Œuvres de Fourier*, Vol. I., p. 507, Paris, 1888, where the integral

$$\int_{-\infty}^{\infty}d\alpha f(\alpha)\int_{-\infty}^{\infty}\frac{dp}{p}\cos(px-p\alpha)(e^{py}+e^{-py})$$

is considered without any mention of convergence; this has probably been often remarked before. The convergency of the integral (123) above is obtained by the insertion of the constant $\int_{-\infty}^{\infty}\int_0^{\infty}F(v)\frac{e^{-u}}{u}du\,dv$ which is also infinite.

from equation (135) we obtain

$$\phi(\alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\frac{\partial \phi(\alpha, v)}{\partial \alpha}\right)_{\alpha=1} \left(\frac{e^{-u}}{u} - \frac{e^{-u(\alpha-1)}}{u} \cos u (v-\beta)\right) du dv \quad \ldots(138)$$

as the formal solution of our flexure problem, when combined with equation (91). The expression for $\phi(\alpha, \beta)$ takes a simpler form when we put $v = \tan w$, $\beta = \tan \psi$.

The value of

$$\int \{x \cos(y\nu) - y \cos(x\nu)\} \phi ds$$

in the torsion equation (94) reduces to the simple expression

$$\frac{1}{m} a^2 \int_{-\infty}^{+\infty} \frac{(\phi_{\alpha=1}) \beta}{(1+\beta^2)^{\frac{m+1}{m}}} d\beta = -\frac{a^2}{2} \int_{-\infty}^{+\infty} \phi_{\alpha=1} d \frac{1}{(1+\beta^2)^{\frac{1}{m}}}$$

$$= -\frac{a^2}{2\pi} \int_{-\infty}^{+\infty} \int_{0}^{\infty} \int_{-\infty}^{+\infty} F(v) \sin u (v-\beta) (1+\beta^2)^{-\frac{1}{m}} d\beta du dv;$$

which completes the formal solution, but is an expression hardly likely to prove of value.

The portion of the integral (136) which concerns u depends essentially on the function $\log((\alpha-1)^2 + (v-\beta)^2)$, and it is suggested that a variant formal solution would be obtained by taking

$$\phi(\alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(v) \log\{(\alpha-1)^2 + (v-\beta)^2\} dv \quad \ldots\ldots\ldots\ldots(139).$$

Assuming convergence, we can easily show that this integral satisfies the differential equation

$$\frac{\partial^2 \phi}{\partial \alpha^2} + \frac{\partial^2 \phi}{\partial \beta^2} = 0,$$

and that

$$\left(\frac{\partial \phi}{\partial \alpha}\right)_{\alpha=1} = F(\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(140),$$

so that as before we can take as solution

$$\phi(\alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\partial \phi(\alpha, v)}{\partial \alpha}\right)_{\alpha=1} \log\{(\alpha-1)^2 + (v-\beta)^2\} dv \quad \ldots\ldots(141).$$

The question of the convergence of the integrals used in this discussion clearly requires further investigation. At first sight it might appear that the integral in (139) is divergent, seeing that logarithmic term contains $(v-\beta)^2$, but it is to be remembered that $F(v)$ is in our case an *odd* function of v, so that it is not decided that

$$\int_{-\infty}^{\infty} F(v) \log\{(\alpha-1)^2 + (v-\beta)^2\} dv = \operatorname*{Lt}_{v \to \infty} \int_{-v}^{v} F(v) \log\{(\alpha-1)^2 + (v-\beta)^2\} dv$$

is infinite. There is further to be decided the uniform convergence of the integrals when $\beta = \pm\infty$. These points are left over for future discussion.

The crude formal results given in the latter part of this section may prove on further investigation to be unworkable, or at least unprofitable, and are published here only by way of indicating the lines along which it is intended to continue the work when opportunity offers.

Conclusion.

To the engineer and especially to the designer of aeroplanes the chief interest of this work will lie in its application to the problem of the change of the 'angle of attack' that arises in the blades of a propeller rotating at high speed, and it may be well to state here briefly the results that seem to affect that problem. It is first of all to be emphasized that the solutions given in the memoir are solutions of problems of much less generality than those of an actual propeller-blade and that the results can therefore be regarded only as first approximations to what is wanted. Thus the beams considered are *uniform* in cross-section throughout their length and that cross-section is in every case symmetrical about an axis which is perpendicular to the plane of flexure. In beams of cross-sections having no axis of symmetry the torsion effect would presumably be of greater importance*. Moreover the loading is not continuous as it is in an actual propeller but consists of a single load acting at the end of the beam. With these reservations we can state the result:

For a beam of isotropic material and of uniform uni-symmetrical cross-section of shape resembling the cross-section of a propeller-blade, flexed by a load of weight W applied at one end in a direction perpendicular to the axis of symmetry, the torsion per unit length due to the unbalanced shear-couple may be taken as approximately † equal to $\cdot 28 \frac{Wa}{EI}$, when E is the Young's modulus of the material, a is the breadth of the beam and I is the second moment of the cross-section about the axis of symmetry. The direction of the torsion is such that the thicker side of the beam twists *upwards* with a downwards flexing force.

In beams of cross-section resembling the cross-section of a propeller-blade the torsion due to anticlastic curvature is in the opposite sense to the shear torsion and roughly about half its magnitude.

If the material of the beam be aeolotropic we have seen that the torsion due to the unbalanced shear-couple may be increased from twice to six times the isotropic value. This increase is largely determined by the ratio E/μ, where E is the longitudinal stretch-modulus and μ is the transverse slide-modulus parallel to the direction of loading, but it is far from being directly proportional to this ratio.

When, however, wood is the material, each particular beam would appear to have its own amount of torsion, the asymmetry of the elastic properties of the material being apparently more effective than the asymmetry of shape. To the designer of aeroplane propellers this should be of great interest for, as mentioned above‡ after

* The amount of torsion will, of course, be a maximum when the load is perpendicular to the axis of symmetry. If it make an angle ψ with it, only the component $W \sin \psi$ produces torsion.

† For the calculations on which this estimate is based see §§ 10 and 13.

‡ In § 19.

the description of an experiment which brought out the importance of this asymmetry of texture, propellers might be constructed of layers of wood selected and arranged in such a way as to minimise the torsion which arises from flexure.

Apart from the propeller-blade approximation the most interesting case theoretically is that of the split pipe in which the torsion effect is so pronounced. We do not know of any instance in practical engineering in which this is of much importance although such may well exist, but it is conceivable that some practical value could be found for the phenomenon in such a pipe, for example as an economical form of cantilever weighing machine for heavy loads or in circumstances in which the ordinary methods would be inconvenient. It might also be used for light loads as a letter-weighing physical toy. Such value arises from the fact that the angle of torsion is always proportional to the suspended load and is very large if the wall of the pipe be thin.

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