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AERODYNAMICS

Photograph Showing Flow of Air Round a Cylinder in Motion.

[*Frontispiece.*

AERODYNAMICS

CONSTITUTING THE FIRST VOLUME OF A COMPLETE WORK ON AERIAL FLIGHT

BY

F. W. LANCHESTER

With Appendices on the Velocity and Momentum of Sound Waves, on the Theory of Soaring Flight, etc

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PREFACE.

THE problems that arise in connection with the study of Aerial Flight are so numerous and of so diverse a character that, except for their relation to the title subject, they would scarcely find place in one volume. In the present work an attempt is made, it is believed for the first time, to treat the classification of the phenomena associated with the study of Flight on a comprehensive and scientific basis.

The origin of the present work may be said to date from some experiments carried out in the year 1894. These experiments, which were primarily directed as a test of certain theoretical views which the author then advanced, resulted in the production of flying models of remarkable stability, whose equilibrium was not destroyed by an ordinary gale of wind.

As originally formulated the theory was incomplete and in many ways imperfect, but it has been developed from time to time during the last twelve years to an extent that to-day renders the approximately correct proportioning of an *aerodrome*[1] a matter of straightforward calculation.

The author has found the question of publication one of some difficulty. At first it was intended to arrange and publish the investigations simply in order of date, theoretical work being accompanied so far as possible by appropriate experimental

[1] A word derived from the Greek, ἀερο-δρόμος (lit. "*traversing the air*" or "*an air-runner*"), proposed by the late Prof. Langley to denote a gliding appliance or flying machine; hence also *aerodromics*, the science specifically involved in the problems connected with *free flight*. The word *aerodrome* has been grossly misapplied by Continental writers to denote a balloon shed. The author considers that from its derivation the word *aerodromics* may be given a more comprehensive meaning than that originally proposed, perhaps even to include both the *aerodynamics* and *aerodonetics* of flight. The question is merely one of terminology. (Compare Glossary, p. 393.)

demonstration. It soon became evident that there were considerable *lacunae*, and these were filled by subsequent investigations, the scope of the work being greatly extended. Finally it was decided to make the publication a complete treatise on *Aerial Flight*, the main classification being as follows:—

Vol. I. *Aerodynamics*, relating to the theory of aerodynamic support and the resistance of bodies in motion in a fluid.

Vol. II. *Aerodonetics*[1] or *Aerodromics*, dealing with the forms of natural flight path, with the questions of equilibrium and stability in flight, and with the phenomenon of "soaring."

So far as has been found possible the work has been modelled on *non-mathematical lines*. The commonly distinctive feature of a modern *mathematical* treatise, in any branch of physics, is that the investigation of any problem is initially conducted on the widest and most comprehensive basis, equations being first obtained in their most general form, the simpler and more obvious cases being allowed to follow naturally, the greater including the less. The reader who is only moderately equipped with mathematical knowledge is thus frequently at a loss to comprehend the initial stages of the argument, and so has no great chance of fully appreciating the conclusions.

It is impossible, in connection with the present subject, to avoid the frequent use of mathematical reasoning, and occasionally the non-mathematical reader may find himself out of his depth. The author has endeavoured to minimise any difficulty on this score by dealing initially with the simpler cases and afterwards working up to the more general solutions; and further by the careful statement of all propositions apart from mathematical expression, and by the re-statement of conclusions in non-mathematical language. Wherever appropriate, numerical examples are given in order to more completely elucidate the methods employed and the results attained.[2]

[1] Derived from the Greek, ἀεροδόνητος (lit. "*tossed in mid-air*," "*soaring*").

[2] A passage occurs in the preface to Poynting and Thomson's "Sound" that may be quoted as being to the point:—

"Even for the reader who is mathematically trained, there is some advantage in the study of elementary methods compensating for their cumbrous

Whenever the author has consciously derived assistance from the work of previous investigators, due acknowledgment has been made; the present work is, however, in the main, a connected series of personal investigations. Should the author inadvertently have put forward as new, results that have been previously published or methods that have been previously employed, he can at least claim in mitigation of the offence that very many of the present investigations were actually done more than ten years ago; the work has only been withheld to the present date in order that publication might take the form of a complete and connected account of the mechanical principles of flight such as could be the better understood by, and be of the greater service to, the Scientific and Engineering World.

In offering to the public the first instalment of the present work, the author desires to record his conviction that the time is near when the study of Aerial Flight will take its place as one of the foremost of the applied sciences, one of which the underlying principles furnish some of the most beautiful and fascinating problems in the whole domain of practical dynamics.

In order that real and consistent progress should be made in Aerodynamics and Aerodonetics, apart from their application in the engineering problem of mechanical flight, it is desirable, if not essential, that provision should be made for the special and systematic study of these subjects in one or more of our great Universities, provision in the form of an adequate endowment with proper scope for its employment under an effective and enlightened administration.

The importance of this matter entitles it to rank almost as a National obligation; for the country in which facilities are given for the proper theoretical and experimental study of flight will inevitably find itself in the best position to take the lead in its application and practical development. That this must be

form. They bring before us more evidently the points at which the various assumptions are made, and they render more prominent the conditions under which the theory holds good."

considered a vital question from a National point of view is beyond dispute; under the conditions of the near future the command of the air must become at least as essential to the safety of the Empire as will be our continued supremacy on the high seas.

The present volume deals exclusively with the Aerodynamics of Flight; the arrangement of this section is as follows:—

Chapters I., II. and III. are devoted to the preliminary exposition of the underlying principles of fluid dynamics, examined from different points of view. Chapter I. is of an introductory character, and includes a discussion as to the nature of *fluid resistance*, the theory of the *Newtonian medium*, and a preliminary examination of the questions of *discontinuous motion* and *streamline form*. Chapter II. is devoted to the consideration of *viscosity* and *skin-friction*, the argument being largely founded on dimensional theory; and Chapter III. consists in the main of an account of the *Eulerian hydrodynamic theory*, in which the mathematical demonstrations are in general taken for granted;[1] this chapter also includes some further discussion of the phenomenon of *discontinuous flow* and a review of the controversy relating to same.

Chapter IV. consists in most part of an investigation on *peripteral motion*,[2] dating from the year 1894–5 and offered to the Physical Society of London in the year 1897, but rejected.[3]

[1] The reader is referred to "Hydrodynamics" (Horace Lamb, Cambridge University Press) for the complete mathematical treatment: a work to which the author desires to acknowledge his indebtedness.

[2] A term proposed and employed by the author to denote the type of fluid motion generated in the vicinity of a bird's wing, or the supporting member of an aerodrome essential to its supporting function (lit. "round about the wing," Gr. περι and πτερόν). The term has an architectural signification which can by no possibility clash with its present usage.

[3] The rejection of this paper was probably due to an unfortunate selection of the readers to whom it was submitted. The names of the Society's readers are not disclosed, but from the wording of the reports (which the author is not at liberty to quote), it would seem that the recognised application of the Newtonian method (as in the theory of propulsion) was a thing unknown to them.

The hydrodynamic interpretation included in the present work has been added subsequently, and the latter portion of the original paper has been revised and rewritten on the more secure basis thus afforded.

Chapters V. and VI. constitute a *résumé* of that which is known concerning the *aeroplane* treated both from a theoretical and experimental standpoint.

Chapters VII. and VIII. present, for the first time, a series of investigations made by the author (dating from 1894, 1898, and 1902, but not previously published) of the principles governing the *economics of flight*, and their application in the correct proportioning of the supporting member; these investigations are based on the peripteral theory of Chapter IV. aided by a hypothesis, being in the main an adaptation of Newtonian method.[1]

Chapter IX. includes, with a discussion on the elementary theory of propulsion, an original investigation on the theory of the *screw propeller* founded on the peripteral theory of Chapters IV., VII., and VIII. This theory leads to results that are in remarkable accord with experience, and enables a useful series of rules to be laid down as a guide to design; applied to the marine propeller, the theory gives a form quite in harmony with modern practice. The chapter concludes with a dissertation on the subject of the *expenditure of power in flight*.

Chapter X., with which the present volume concludes, is of the character of an appendix, being an account of the more important of the experimental researches in aerodynamics published to date, and to which references have been made in the body of the work. This chapter also includes an account of some hitherto

[1] The essentially Newtonian character of all methods based on the principle of the direct communication of momentum, in hydrodynamics, is not so widely recognised as it ought to be. Thus the Rankine-Froude theory of propulsion is a simple and legitimate application of the Newtonian theory (see Chap. IX.). Newton was careful to specify the nature of the *medium* essential to the rigid application of his method (prop. xxxiv., Book II., *Enunciation*); subsequent writers have unfortunately not been so careful, and error has resulted.

unpublished experiments by the author, and some criticism of the conclusions formulated by earlier investigators.

A few terminological innovations have been made at one time and another, as necessity has arisen. New words, or words bearing a special or restricted meaning, are given in the *glossary* following Chapter X., in addition to the usual footnote references.

Numerical work has been done by the aid of an ordinary 25 c.m. slide rule, with a liability to error of about $\frac{1}{5}$th of 1 per cent., an amount which is quite unimportant.

The author desires to express his thanks to Mr. P. L. Gray in connection with the preparation of the present volume for the Press, in particular for his most welcome assistance in the examination and correction of the proof sheets.

BIRMINGHAM,
October, 1907.

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AERODYNAMICS.

CHAPTER I.

FLUID RESISTANCE AND ITS ASSOCIATED PHENOMENA.

§ 1. **Introductory.**—A body in motion through a fluid of any kind, whether liquid or gaseous, experiences resistance, and work has to be done in its propulsion.

Such resistance is due to two clearly distinct causes, the independent nature of which may be illustrated by considering a few commonplace instances.

If a piece of cardboard be moved briskly through the air, the resistance, though quite sensible, is very much less than that experienced when a similar movement is attempted under water. In this case the difference is evidently due to the very much greater density of water, which at 10° C. is 800 times that of air. If now we similarly compare water with any ordinary grade of heavy lubricating oil, or with common treacle, we again find a great difference, but this time the density is approximately the same, and we recognise that the resistance is due to an entirely different cause; a certain *stickiness* of the medium, otherwise *viscosity*. All fluids are *viscous* to a greater or lesser degree; the viscosity of water is small, that of air still less, whilst lubricating oil and treacle are highly viscous substances.

Now just as in the study of ordinary *mechanics* it is found expedient initially to neglect the effects of friction, so, in connection with the present subject, we can afford to ignore the *fluid friction* to which the viscosity of the fluid gives rise, and in the first instance deal with resistance as a function of density alone.

The analogy here suggested is not complete. It frequently happens in the case of fluids that the effects of viscosity have to be taken into account as part of the general dynamic system; consequently it is sometimes necessary to devote some attention to these effects even in the preliminary discussion.

The question of *compressibility* is one on which also it is desirable to have some convention. It is popularly supposed that whereas gases are *compressible*, liquids are virtually *incompressible*; no broad distinction of this kind is justified. The criterion of compressibility in fluid dynamics involves the relative density of the fluid, and on this basis air is only *about* eighteen times as compressible as water, the ratio of the velocity of sound in water and air being approximately in the proportion of $\sqrt{18} : 1$. It is shown later that the influence of compressibility only becomes manifest as the velocity of motion approaches the velocity of sound in the fluid in question, or if the pressures developed involve a serious change in density.

The velocities and pressures ordinarily involved in aërial flight are such as will justify the initial assumption that the air is *incompressible*, that is to say, that the *influence of its compressibility is negligible*. The possibility of error resulting from this hypothesis will be considered subsequently. (Appendix I.)

§ 2. **Two Methods.**—There are two ways in which problems in fluid dynamics may be approached: (1) By the method of the *Newtonian medium*; this, though of great service in certain special cases, is not strictly applicable to real fluids. (2) By the methods of Euler and Lagrange, by which complete equations of motion are obtained, defining the flow of the fluid in the three co-ordinate dimensions of space. This is the method employed in works on analytical hydrodynamics, and discussed in Chap. III. of the present work.

The basis of the Newtonian method is found in the principle of the *conservation of momentum*, which may be taken as corollary to the third law of motion as written: *When force acts on a body* *the momentum generated in unit time is proportional to the force.* This method is best studied in connection with a hypothetical medium suggested by Newton, on which he based several of the problems in the "Principia." This medium is defined as consisting of a large number of material particles, equally distributed in space, having no sensible magnitude, but possessing mass; the particles are not supposed to act upon or be connected to each other in any way. Bodies traversing a region filled with this *medium* experience a resistance which is proportional to the momentum communicated per second, and is a quantity that can be calculated mathematically, provided that the velocity of the body and the density of the medium be known, and the surface *in presentation* of the body be defined.

The employment of this method and its deficiencies in the case of a real fluid are illustrated in the case of the *normal plane* (Chap. V.), where it is found to give a considerably greater pressure value than actually obtains; the general form of the pressure law is, however, in approximate accord.

§ 3. The Newtonian Method.—Employing absolute units, let F = the resistance, let m = mass acted upon during time t, and v the velocity in the line of motion imparted to the mass m; then the fundamental equation is: $F = \frac{m\,v}{t}$.

Now so long as we are dealing with a simple body of mass m, and imparting to it a velocity v, the above equation is merely a statement of the *law of motion* cited, any constant being eliminated by the fact that we are employing absolute units. The equation, however, holds good whatever the number of parts into which the mass be divided, and however the velocities of the different parts vary amongst themselves. In this case the expression may be written: $F = \frac{\Sigma\,(m\,v)}{t}$. The proof is as follows:—

Let us suppose that the mass acted on per second be divided into n parts, and that each part be acted on by the force F for $1/n$ th of a second. Then the momentum communicated to each part $= F/n$, and the total momentum per second $= n\frac{F}{n} = F$, which holds good when the number of parts becomes infinite and the communication of momentum continuous. And since the communication of momentum for each of the periods of $1/n$ th second is independent of the masses of the individual parts, it is in nowise essential that the n parts are of equal mass; consequently the velocities acquired by the different parts may vary amongst themselves to any extent, without thereby affecting the total quantity of momentum communicated.

This principle in its application to fluid dynamics has sometimes been termed the *Doctrine of the Continuous Communication of Momentum.*

§ 4. Application of the Newtonian Method in the Case of the Normal Plane.—To illustrate the method in the case of the normal plane in motion in a region supposed filled with the medium of Newton, we must first define the mode in which the surface of plane the imparts velocity to the constituent particles.[1] If, on the one hand, the body and the particles be supposed perfectly elastic, then the particles on colliding with the surface will bounce off with a velocity equal and opposite (relatively) to that with which they strike; that is to say, if V be the velocity of the plane, and v be the velocity given to the projected particles, v will be double of V. If, on the other hand, we suppose that the plane is inelastic, and that it *eats up* or *absorbs* the particles on impact, then the velocity imparted to them will be equal to that of the plane, or, $v = V$. It is thus of little consequence which hypothesis we take, the one will give a result exactly twice as great as the other. We will select the second hypothesis, which will give the lesser value of the two.

Let us assume the medium as of the same density as air at

[1] Compare "Principia," prop. xxxv., Book II.

14° C., and 760 m.m. pressure, that is to say, let one cubic foot contain $\frac{1}{13}$th lb. mass. Let P represent the pressure per square foot, that is, the total force F divided by the area of the plane. Then the mass dealt with per second to develop a force P will be $\frac{V}{13}$, and the velocity v being equal to V, we have :—

$$\text{Momentum per second} = P = \frac{V^2}{13}.$$

But P here is in absolute units, *poundals*. Reducing to pounds, we have :—

$$\text{Pressure} = \frac{V^2}{13\,g} = \frac{V^2}{13 \times 32{\cdot}2} = \frac{V^2}{420}$$ approximately. If the velocity be expressed in *miles per hour*, this becomes $\frac{V^2}{200}$ (nearly). This may be recognised at once as a result often given in text books as the pressure-velocity equation for *air*, and is tacitly put forward as if founded on experiment. It is approximately 50 per cent. higher than the true value.

If, instead of introducing a value for the density, we denote this by ρ, the expression (absolute units) is: $P = \rho V^2$; the experimental value is, in the case of air, $P = {\cdot}7\,\rho V^2$, or, in the case of water, $P = {\cdot}55\,\rho V^2$, as ascertained for flat plates of compact outline. (See Chap. V.)

§ 5. Deficiency of the Newtonian Method.—It is evident from the foregoing that the theory of the Newtonian medium is capable of giving results within measure of the truth, when applied to real fluids. The degree of accuracy varies with the circumstances, and the author will now endeavour to point out the reasons why, and the manner in which, the method fails, and indicate the circumstances under which the Newtonian theory is applicable and those under which it is not.

At the outset it may be set down that any defect in the theory is due, not to any want of exactitude in the fundamental theory —this rests definitely on the third law of motion and is absolute—but rather to the difficulty and uncertainty as to its manner of application in the case of real fluids.

The nature of this difficulty is clearly demonstrated by the following proposition :—

When a body, propelled through an incompressible fluid, contained within a fixed enclosure, experiences resistance to its motion, the force exerted by the body on the fluid does not impart momentum to the fluid, but is transmitted instantly to the confines of the fluid however remote, and is wholly borne by its boundary surfaces.

Let us suppose (Fig. 1) a body which we will take to be a normal plane C, acted upon by a force F in an enclosure A,

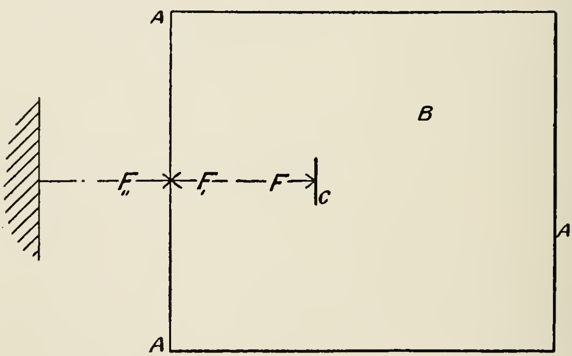

FIG. 1.

filled with fluid B. The enclosure may be supposed as large as we please, or, in the limit, infinite in its dimensions.

Then the condition that the enclosure is fixed denotes that the force F applied *to* the plane is applied *from* the walls of the enclosure; for, if we suppose it applied *from without* we can resolve the force into a force acting between the plane and the enclosure $F\,F_{\prime}$, and a force of equal magnitude acting *from without* on the enclosure $F_{\prime\prime}$, and since the enclosure is fixed the latter can have no effect.

Now since the fluid is incompressible its density is constant and uniform, therefore the mass centre of the contents cannot move relatively to the enclosure; and the enclosure itself is fixed, consequently the fluid in sum does not receive momentum.

If, in place of a real fluid possessing continuity, we had supposed the enclosure filled with the medium of Newton, then momentum would have been communicated in sum to the particles of the medium, and the resistance could be calculated in the manner already demonstrated.

If we take away the condition that the enclosure is *fixed* and suppose the force applied *from without,* then the problem is not essentially altered, for though the external force F will now impart momentum to the system *en bloc*, its action in this respect has no relation to phenomena in the interior, and does not provide any data for the determination of the pressure-velocity relation.

The supposition that the body is a *plane* evades any question relative to the density of the body itself, and thus simplifies the argument. This question could also be eliminated by supposing the body to possess the same density as the surrounding fluid; in any case a force applied to the body to overcome its inertia is a matter external to, and without influence on, the conditions.

The foregoing proposition cannot depend in any way upon the viscosity or otherwise of the fluid; the existence of viscosity can affect the mode of transmission of the force and the velocity of the body that accompanies its transmission, but can have no influence on the total force transmitted.

It is thus apparent that no momentum is imparted to an actual fluid in the sense that it is imparted to the Newtonian medium, and this is the real cause of the difficulty in the application of the Newtonian method.

The principle here demonstrated is referred to in the present work as the "Principle of No Momentum."

§ 6. Illustrations of the Principle of No Momentum.—The foregoing proposition is of moment in connection with several problems in fluid dynamics, and presents the subject in an aspect that is somewhat unfamiliar. Its import may be pointed by the following illustrations.

A body of apparent weight F, falls uniformly through a column of *inviscid* or frictionless fluid, contained in a vertical cylindrical or prismatic vessel, open at the top. Then the weight of the body (F) will be carried as additional pressure on the base of the vessel during the whole time of the descent; and if the vessel be tall and narrow the additional pressure will be approximately uniform and equivalent to an additional "head"; if it be wide, so that the walls are remote from the body, then the distribution on the pressure area will not be uniform, but will be greatest at the point vertically beneath the body, and less at points more remote. If the fluid possess viscosity, the whole of the force F may not reach the base of the vessel, but will in part be borne by its walls, but the total force carried by walls and base will in any case be equal to F.

In the above illustration there is nothing that is strikingly unfamiliar. If we suppose the vessel to be an ordinary jar of liquid, placed in the scale pan of a balance, there is a certain obviousness in the fact stated; the weight of the whole will be just the same whether the weight rests inert at the base of the jar, or whether it be falling uniformly through the fluid. When, however, the principle is applied to bodies aerodynamically supported in the free atmosphere the matter is not so self-evident; here, for example, we find that the weight of a parachutist is borne by the earth's surface almost from the moment he leaves the car, and his presence overhead, or the presence of a passing flight of birds, could be detected barometrically if we possessed an instrument of sufficient delicacy.

§ 7. Transmission of Force.—Comparison of Fluid and Solid.—We know that we may look upon a solid in stress as communicating momentum, since it transmits force, but a distinction must be drawn. When the flow of momentum is equal and opposite, as in the case of a solid in stress, there is no displacement of matter, and it is only when there is a displacement of matter that the Newtonian method can be applied. The case of a gas under pressure is, according to the kinetic theory, an example of the actual communication of momentum, and its pressure and the mean velocity can be correlated on the Newtonian principle; but once lose sight of the transference of matter (molecular motion), and we can only assert that the gas is exerting and transmitting force.

As a whole, the fluid, in the previous section, does not gain or lose momentum any more than does a cast-iron pillar supporting a load. The stress is transmitted in part by viscosity and in part dynamically; the part that is transmitted dynamically is transmitted by an actual transference of momentum from certain parts of the fluid to certain other parts; but this we cannot follow without equating the motions of the fluid throughout the whole of the enclosed space. The manner in which a portion of the stress is transmitted by viscosity may be compared, if we adopt a view put forward by Poisson and Maxwell, to its transference by a solid continually giving way in shear; or, on the other hand, if the fluid is gaseous, we may, on the kinetic theory, regard the viscous resistance as of purely dynamic origin, but belonging to a system quite apart from that of the aerodynamic disturbance.

§ 8. When the Newtonian Method is Applicable.—In the case of the Newtonian medium the quantity of matter dealt with, and momentum imparted per unit time, are defined quantities; but in the real fluid it has been shown that the motion produced is a circulation of the fluid not accompanied by any total change of momentum, and although parts of the fluid receive momentum in the direction of the applied force, other parts receive momentum in the opposite direction. In spite of this difficulty, there are certain cases in which the principle of the continuous communication of momentum can be applied. A most striking example is to be found in the theory of marine propulsion founded by Rankine and Froude.

According to this theory the propeller (whether screw, paddle, or jet propulsion be employed) is taken as operating on a certain mass of fluid per second, to which it imparts a certain sternward velocity. It is assumed that the momentum per second so imparted constitutes and accounts for the whole propulsive force, an assumption that under practical conditions is doubtless very close to the truth. In the case of the screw propeller the mass of fluid per second is calculated from the volume of the cylindrical body of water defined by the track of an imaginary circle drawn through the tips of the blades ; in other forms of propulsion similar approximate methods of assessment are adopted.

The sternward velocity imparted to the fluid by the propeller is, under proper conditions, small in comparison with the velocity of travel, so that the lines of flow are not radically altered, and instead of a circulation such as arises in the case of a normal plane, there is merely a slight contraction of the stream at the region in which the propeller operates, and a trifling readjustment of the surrounding lines of flow to suit.

In general it would appear that the Newtonian method is applicable in cases where the volume of the fluid handled is great, but where the impressed velocity is small in comparison with the velocity of motion, and where there are well-defined conditions on which to compute the amount of fluid dealt with per second, it is found to be entirely deficient in dealing with the resistance of bodies of smooth contour, or "streamline form," such as may now be discussed.

§ 9. On Streamline Form.—When a body of fish-shaped or *icthyoid* form travels in the direction of its axis through a frictionless fluid there is no disturbance left in its wake. Now we have seen that in any case the fluid as a whole receives no momentum, so that it is perhaps scarcely legitimate to argue that there is no resistance *because* there is no communication of momentum, although this is a common statement.[1] It is clear,

[1] This somewhat academic objection would cease to apply if any means could be found to properly define the *idea* which undoubtedly is conveyed to the miud by the argument in question.

however, that if there is no residuary disturbance there is no necessary expenditure of energy, and this equally implies that the resistance is *nil*.

The fluid in the vicinity of a streamline body is of necessity in a state of motion and contains energy, but this energy is *conserved*, and accompanies the body in its travels, just as in the case of the energy of a wave. It adds to the kinetic energy of the body in motion just as would an addition to its mass.

According to the mathematical theory of Euler and Lagrange, all bodies are of streamline form. This conclusion, which would otherwise constitute a *reductio ad absurdum*, is usually explained on the ground that the fluid of theory is *inviscid*, whereas real fluids

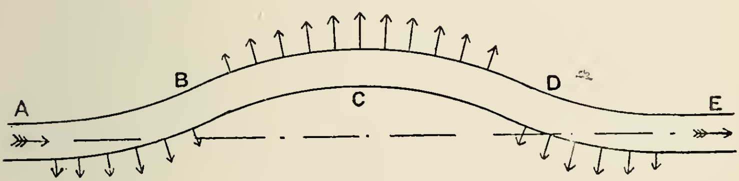

Fig. 2.

possess viscosity. It is questionable whether this explanation alone is adequate.

§ 10. Froude's Demonstration.—An explanation of the manner of the conservation of kinetic energy, in the case of a streamline body, has been given by the late Mr. W. Froude.

Referring to Fig. 2, *A*, *B*, *C*, *D*, *E*, represents a bent pipe, through which a fluid is supposed to flow, say in the direction of the lettering, the direction at *A* and at *E* being in the same straight line; it is assumed that the fluid is frictionless. Now so long as the bends in the pipe are sufficiently gradual, we know that they cause no sensible resistance to the motion of the fluid. We have excluded viscous resistance by hypothesis, and if the areas at the points *A* and *E* are equal there is no change in the kinetic energy. Moreover, the sectional area of the pipe between the points A and E may vary so long as the variations are gradual; change of pressure will accompany change of area on well-known hydrodynamic principles, but no net resistance is introduced; consequently the motion of the fluid through the pipe does not involve any energy expenditure whatever.

Let us now examine the forces exerted by the fluid on different portions of the pipe in its passage. The path of the particles of fluid in the length between the points A and B is such as denotes *upward acceleration*, and consequently the fluid here must be acted on by an upward force supplied by the walls of the pipe, and the reaction exerted by the fluid on the pipe is equal and opposite. A shorter way is to regard this reaction as the centrifugal component of the curvilinear path of the flow, and as such it may be indicated by arrows as in the figure.

By assuming the bends in the pipe to be equal and a uniform velocity throughout, it follows that these centrifugal components exactly balance one another, each to each, and the pipe has no unbalanced force tending to push it in one direction or the other. The argument may be found presented in this form in White's "Naval Architecture." The same net result follows, no matter what the exact form of the bends, or whether or no the velocity is uniform, provided the bends are smooth and the cross-section (and therefore the velocity) is the same at E as at A, for under these circumstances the pressure at A will be the same as at E, the applied forces thus being balanced, and there will be no momentum communicated by the fluid in its passage.

When a streamline body travels through a fluid the lines of flow may be regarded as passing round it as if conveyed by a number of pipes as in Fig. 2. It is convenient, and it in nowise alters the problem, to look upon the body as stationary in an infinite stream of fluid (Fig. 3); we are then able to show clearly the lines of flow relatively to the surface of the body. Now let us take first the fluid stream that skirts the surface itself, and let us suppose this included between the walls of an imaginary pipe, then forces will be developed in a manner represented in Fig. 2, and these forces may be taken as acting on the surface of the body. It is not necessary to suppose that there is actual *tension* in the fluid, as might be imagined from Fig. 3, where the forces act outward from the body, this is obviated by the general hydrostatic pressure that obtains in the region; the forces as drawn are those supplied by the motion of the fluid, and can be looked upon as superposed on those due to the static pressure.

If, similarly, we deal with the next surrounding layer of fluid, we find that the pressure to which it gives rise acts to reinforce that of the layer underneath (*i.e.*, nearer the body), and so on, just as in hydrostatics the pressure is continually increased by

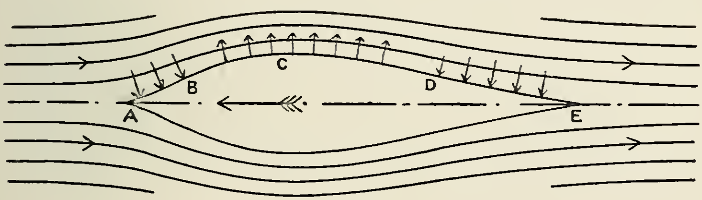

FIG. 3.

the addition of superincumbent layers of fluid, and thus we find that the body is subjected to increased pressure acting on its front and rear, and diminished pressure over its middle portion. Now it has been shown, in the case of the pipe, that the algebraic sum of all forces in the line of motion is zero, so that in the stream-line body the sum of the forces produced by the pressure on its surfaces will be zero, that is to say, it will experience no resistance in its motion through the fluid.

It may be taken as corollary to the above, that in a viscous fluid the resistance of a body of streamline form will be represented *approximately* by the tangential resistance of its exposed area as determined for a flat plate of the same general proportions. This is the form of allowance suggested by Froude; a more elaborate and accurate method has been given by Rankine, in which allowance is made for the variation in the velocity of the fluid at different points on the surface of the body. Neither of these methods includes any allowance for viscous loss owing to the *distortion* of the fluid in the vicinity of the body.

§ 11. The Transference of Energy by the Body.—It is of interest to examine the question of the transference of energy through the streamline body itself from one part of the fluid to the other. For the purpose of reference the different portions of the body have been named as in Fig. 4, the *head*, the *shoulder*, the *buttock*, and the *tail*, the head and shoulder together being termed (as in naval architecture) the *entrance*, and the buttock and tail the *run*. The dividing line between the entrance and run is situated at the point of maximum section, and the dividing line between the

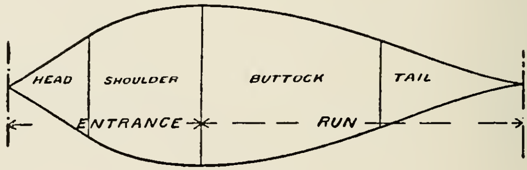

FIG. 4.

head and shoulder on the one hand, and between the buttock and tail on the other, is the line on the surface of the body at which the pressure is that of the hydrostatic "head."

Now, as the body advances, the *head*, being subject to pressure in excess of that due to the hydrostatic "head," is therefore doing work on the fluid; that is to say, *transmitting energy to the fluid*; the *shoulder* also advancing towards the fluid is subject to pressure less than that due to hydrostatic head, and is consequently receiving energy from the fluid; the *buttock*, which is *receding from* the fluid, is also a region of *minus* pressure and so *does work* on the fluid; and lastly, the *tail* is receding under excess pressure and so *receives energy*. We thus see that there are two regions, the head and buttock, that give up energy continuously to the fluid, and two regions, the shoulder and tail, that continuously receive it back again. The condition of perfect streamline motion is that the *energy account shall balance.*

§ 12. Need for Hydrostatic Pressure,—Cavitation.—The motion impressed on the fluid by the pressure region of the head is *compulsory*, unless (as may happen in the case of a navigable balloon) deformation of the envelope can take place. The motion impressed by the shoulder, on the contrary, depends upon hydrostatic pressure, for otherwise there is no obligation on the part of the fluid to follow the surface of the body; hydrostatic pressure is necessary to prevent the formation of a void. The pressure measured from the real zero must everywhere be positive, otherwise the fluid will become *discontinuous* and cease to follow the surface. This is a difficulty that has been actually experienced in connection with screw propellers, and termed *cavitation.*

§ 13. The Motion in the Fluid.—It has been shown that the *head* of a streamline form is surrounded by a region of increased pressure. Consequently the fluid as it approaches this region will have its velocity reduced, and the streamlines will widen out, as shown in Fig. 3 (see also Figs. 42, 44, 45, etc.). This behaviour of the fluid illustrates a point of considerable importance, which is frequently overlooked. Whenever a body is moving in a fluid, its influence becomes sensible considerably in advance of the position it happens to occupy at any instant. The particles of fluid commence to adjust themselves to the impending change with just as much certainty as if the body acted directly on the distant particles through some independent agency, and when the body itself arrives on the scene the motion of the fluid is already conformable to its surfaces. There is no *impact*, as is the case with the Newtonian medium, and the pressure distribution is more often than not quite different from what might be predicted on the Newtonian basis. This behaviour of a fluid is due to its *continuity.*

It follows from elementary considerations that the fluid in the "amidships" region possesses a velocity greater than the general velocity of the fluid (the body, as before, being reckoned stationary). We know that at and about the region C, Fig. 3, the fluid has a less area through which to pass than at other points in the field of flow. It is in sum less than the normal area of the stream by the area of cross-section of the body at the point chosen. But the field of flow is made up of a vast number of tubes of flow, so that in general each tube of flow will be contracted to a greater or less extent, the area of section of the tubes being less at points where the area of the body section is greater. We know that a contraction in a tube of flow denotes an increase of velocity.

Thus on the whole the velocity of the fluid is augmented across any *normal* plane that intersects the body itself, but the increase of velocity is not in any sense uniform in its distribution. In fact, towards the extremities of the body, and in its immediate neighbourhood, we have already seen that the motion of the fluid is actually *slower* than the general stream.

The motion of the fluid is examined from a quantitative point of view in a subsequent chapter (Chap. III.), where plottings are given of the hydrodynamic solution in certain cases.

§ 14. A Question of Relative Motion.—The motion of the fluid has so far been considered from the point of view of an observer fixed relatively to the body; it will be found instructive to examine the same motions from the standpoint of the fluid itself, that is to say, to treat the problem literally as a *body moving through a fluid*, instead of as a *fluid in motion* round a fixed body.

It is evident that the difference is merely one of relative motion. The problems are identical: we require to consider the motions as plotted on co-ordinates belonging to the fluid instead of co-ordinates fixed to the body itself. The relation of the *streamlines* (which we have so far discussed) to the paths of motion (which we now propose to examine) is analogous to that of the cycloid or trochoid to its generating circle.

§ 15. Displacement of the Fluid.—An unfamiliar effect of the passage of a body through a fluid is a *permanent displacement* of the fluid particles. This displacement may be readily demonstrated. If a mass of fluid be moved from any one part of an enclosure to any other part, the enclosure being supposed filled with fluid, there is a circulation of fluid from one side to the other during transit; and if we suppose it to be moved from one side to the other of an imaginary barrier surface, then an equal volume of fluid must cross the same barrier surface in the opposite direction. Now it is of no importance whether the thing we move be a volume of fluid or a solid body, so that when a streamline body passes from one side to the other of a surface composed of adjacent particles of fluid, that surface will undergo displacement in the reverse direction to that in which the body is moving, and the volume included between the positions occupied before and after transit will be equal to the volume of the body itself.

Moreover, since the actual transference of the fluid is due to a circulation from the advancing to the receding side of the body, it will take place principally in the immediate vicinity of the body and less in regions more remote; it is, therefore, immaterial whether the fluid be contained within an enclosure or whether one or more of its confines be free surfaces, provided that *continuity* is maintained, and that the body is not in the vicinity of a free surface.

§ 16. Orbital Motion of the Fluid Particles.—Since the motion of the fluid results in a permanent displacement, the motion of a particle does not, strictly speaking, constitute an *orbit*. It is, however, convenient in cases such as the present to speak of the motion as orbital.

If we could follow the path of a particle along any streamline, and note its change of position relatively to an imaginary particle moving in the path and with the velocity of the undisturbed stream, we should have data for plotting the orbital motion corresponding to the particular streamline chosen. Thus we know that the amplitude of the orbit of any particle, measured at right angles to the direction of flight, is equal to that of the corresponding streamline.

We further know that, in general, the particles have a retrograde motion—that is, their final position is astern of their initial position—also that the maximum retrograde velocity is to be found in the region of maximum amplitude. Beyond this we know that the initial motion of any particle is in the same direction as that of the body, and that this initial motion is greater for particles near the axis of flight than for those far away.

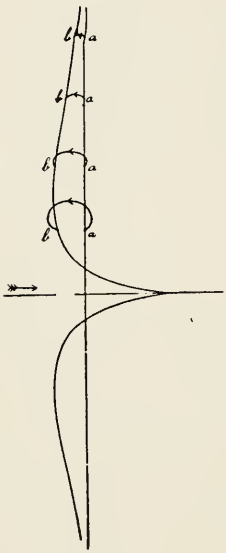

Fig. 5.

Let *b*, *b*, *b*, etc., Fig. 5, represent the final position of a series of particles originally situated in the plane *a*, *a*, *a*; then the orbits of these particles will originate on the plane *a*, *a*, *a*, and terminate on the surface *b*, *b*, *b*, and the motion will be of the character shown.

The form of the surface *b*, *b*, *b*, will be different for different forms of body. It will evidently approach the plane *a*, *a*, *a*, asymptotically, and generally will tend to form a cusp pointing along the axis of flight. The development of this cusp is greatest in cases where the extreme entrance and run are of bluff form, as in the Rankine Oval, Fig. 42, where the point of the cusp is never reached, the surface approaching the axis of flight asymptotically. In reckoning the displacement of the fluid (§ 15), the volume included in the cusped surface forward of the plane *a*, *a*, must be considered negative, since here the fluid is displaced in the same direction as the motion of the body.

§ 17. Orbital Motion and Displacement,—Experimental Demonstration.—The displacement of the fluid and the form of the orbit can be roughly demonstrated by a simple smoke experiment. If a smoke cloud be viewed against a dark background during the passage of a body of streamline form in its vicinity, the retrograde movement of the air is clearly visible. So long as the surface of the body is not too close, the movement is clean and precise, and the general character of the orbit form can be clearly made out; it is found to be, so far as the eye can judge, in complete accord with the foregoing theory. The commencement and end of the orbit, where the motion should be in the same direction as the body, is most difficult to observe, though even this detail is visible if the orbit selected be sufficiently near to the axis of flight. The difficulty here is that the latter part of the orbit is generally lost in consequence of the "frictional wake,"[1] *i.e.*, the current set up by viscous stress in the immediate neighbourhood of the body in motion. In all actual fluids a wake current of this kind is set up, and the displacement surface *b, b, b,* Fig. 5, is obliterated in the neighbourhood of its cusp by a region of turbulence.

§ 18. Orbital Motion,—Rankine's Investigation.—The form of the orbits of the fluid particles has been investigated theoretically for a certain class of body by Rankine (Phil. Trans., 1864).

Rankine closely studied the streamlines of a body of oval form, generated by a certain method from two foci (§ 77), and by calculation arrived at the equation to the orbit motion of the particles. The result gives a curve whose general appearance is given in Fig. 6 (actual plotting), in which the arrows represent the motion of the particle, the direction of motion of the body being from left to right.

Discussing the particular case in which the eccentricity of the oval vanishes, and the form merges into that of a circle, Rankine says,—" . . . The curvature of the orbit varies as the distance

[1] A term used in naval architecture.

of the particle from a line parallel to the axis of X, and midway between that axis and the undisturbed position of the particle. This is the property of the looped or coiled elastic curve; therefore when the water-lines are cyclogenous the orbit of each particle of water forms one loop of an elastic curve." Further, he says—"The particle starts from a, is at first pushed forward, then deviates outwards and turns backwards, moving directly against the motion of the solid body as it passes the point of greatest breadth, as shown. The particle then turns inwards, and ends by following the body, coming to rest at b *in advance of its original position*."

This orbit in some respects resembles that arrived at by the author, but differs in the one very important point that, whereas the author's method gives a *retrograde* displacement of the fluid as the net consequence of the passage of the body, Rankine's conclusion is exactly the contrary.

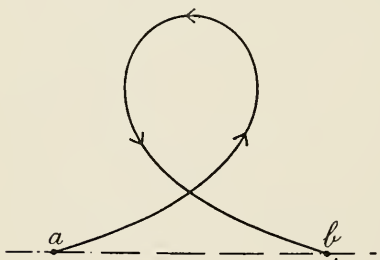

Fig. 6.

As the author's result is capable of experimental verification, it is evident that some subtle error must exist in Rankine's argument, the exact nature of which it is difficult to ascertain.

§ 19. Bodies of Imperfect Streamline Form.—In an actual fluid, bodies of other than streamline form experience resistance apart from that directly due to viscosity.

In the practical shaping of a streamline body it is found essential to avoid corners or sharp curves in the line of flow. Bodies in which due precaution is not taken in this respect offer considerable resistance to motion, and the regions of abrupt curvature give rise to a *discontinuity* in the motion of the fluid. Thus Fig. 7 represents a double cone moving axially, and it will be noticed that the flow has not time to close in round the *run*, as it would do in a properly formed streamline body, but shoots past the sharp edge, as indicated in the figure. The region in the rear of the body, Z, is filled with fluid that does not partake of the general flow, and which is termed *dead-water*.

The resistance experienced by bodies of imperfect form is due to the work done on the fluid, which is not subsequently given back, as is the case with the streamline body. This resistance can be traced to two causes, namely, excess pressure on the surface in presentation and diminished pressure in the dead-water region. The former is of dynamic origin, the energy being expended in directly impressing motion on portions of the fluid; the latter is due to the *entrainment* or viscous drag experienced by the dead-water at the surface bounded by the live stream. It is generally believed that, in a fluid whose viscosity is negligible, the latter cause would be inoperative, the whole resistance being then due to the excess pressure region in front of the body, the dead-water or wake being at approximately the hydrostatic pressure of the fluid.

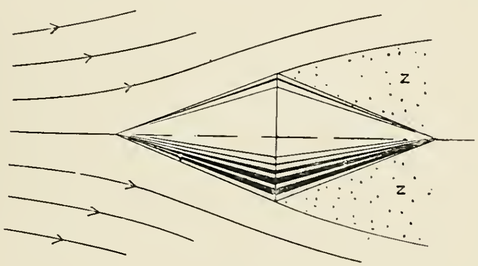

Fig. 7.

The surface separating the live stream and the dead-water constitutes a *discontinuity*, since the *velocity* of the fluid, considered as a function of its *position* in space, is discontinuous. This case is not one of a *physical discontinuity*, such as discussed in § 12, for the region on either side of the surface is filled with the same kind of fluid; it is rather a *kinetic discontinuity*, that is to say a *discontinuity of motion*.

§ 20. The Doctrine of Kinetic Discontinuity.—The theory of kinetic discontinuity is of modern origin, having been introduced and developed by Kirchhoff, Helmholtz, and others, to account for the *phenomenon of resistance* in fluid motion. The analytical theory, based on the hypothesis of continuity, does not in general lead to results in harmony with experience. All bodies, according to the Eulerian theory, are of streamline form, provided that the hydrostatic pressure of the fluid is sufficient to prevent cavitation; we know that in practice this is not the case.

According to the teaching of Helmholtz and Kirchhoff, a kinetic discontinuity can be treated as if it were a physical discontinuity; that is to say, the contents of the dead-water region can be ignored; and this method of treatment is now generally recognised, although not universally so. The controversial aspect of the subject is discussed at length at the conclusion of Chap. III.

The principal objection to the theory of discontinuity is that in an inviscid fluid a surface of discontinuity involves *rotation*, and therefore, by a certain theorem of Lagrange, it is a condition that cannot be generated.[1] A further objection sometimes raised is that such a condition as that contemplated would be unstable, and that the surface of discontinuity, even if formed, would break up into a multitude of eddies. Whether this is the case or not in an inviscid fluid, it is certain that in a fluid possessed of viscosity a surface of discontinuity does commence to break up from the instant of its formation; but as this breaking up does not affect the problem in any important degree, the objection in the case of the inviscid fluid is probably also without weight.

In a real fluid a finite difference of velocity on opposite sides of any surface would betoken an infinite tangential force. Consequently the discontinuity becomes a stratum rather than a surface, and the stratum will either be a region in which a velocity gradient exists (§ 31), or it will become the seat of turbulent motion (§ 37), the latter in all probability.

The conception of the discontinuity as a surface and the method involving this conception are in no way affected by these

[1] Chap. III. §§ 65—71.

considerations. The term *surface of discontinuity* may be looked upon as an abstraction of that which is essential in a somewhat complex phenomenon.

§ 21. Experimental Demonstration of Kinetic Discontinuity.— The reality and importance of the discontinuous type of motion can be demonstrated conclusively by experiment.

In Fig. 8, *a*, *b*, *c*, is a hollow spherical globe in which *d* is a tube arranged to project in the manner shown. An ordinary lamp globe and chimney will be found to answer the purpose

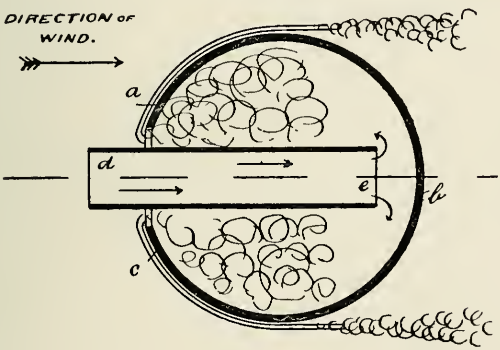

FIG. 8.

the former having one of its apertures closed by a paper disc. The whole is carefully filled with smoke and then moved through the air in a direction from right to left, the relative direction of the air being indicated by the arrow.

It will be found that the air will enter the tube and displace the smoke through the annular aperture. The issuing smoke follows the surface of the sphere in the most approved manner as far as the "equator," but then passes away at a tangent, the stratum of discontinuity, the dead-water region, and the turbulent character of motion, being all clearly manifest. The discontinuity, as may have been anticipated, does not appear as a clean-cut surface; it is marked almost from the commencement, as indicated in the figure, by eddy motion; but when we remember

Fig. 9.

(Negative, for Positive see Frontispiece.)

that, according to the Eulerian theory, the lines of flow should carry the smoke along a symmetrical path to the opposite pole of the sphere, as in Fig. 45 (Chap. III.), the conclusion is plain.

The author has succeeded in photographing the flow round a cylinder in motion in a smoke-laden atmosphere (Fig. 9). In this example it may be noticed that the surface or stratum of discontinuity arises from a line some distance in front of the plane of maximum section; the difference in the behaviour of a cylinder and sphere in this respect is due to the fact that in the former case the lines of flow are cramped laterally, the motion being confined to two dimensions, whereas in the latter case, the motion being in three dimensions, the fluid can "get away" with greater facility. This difference is reflected in the lower coefficient of resistance found experimentally for the sphere than that ascertained for the cylinder. Thus in the experiments of Dines (§ 226) the *pressure per square foot of maximum section* on a $\frac{5}{8}$-in. cylindrical rod was found to be more than double that on a 6-in. sphere, though doubtless the difference in size in the bodies compared may contribute something to the disparity.

The theory of discontinuity also receives support of the most convincing description from the experiments of Hutton, 1788, and Dines, 1889, by which it is shown that the pressure on a *solid hemisphere*, or a *hemispherical cup* (such as used on the Robinson anemometer), both in *spherical presentation*, does not differ from that on a *complete sphere* to an extent that experiment will disclose. This not only disposes of the streamline sphere of mathematical conception, but proves at the same time the approximate constancy of wake pressure under variation of rear body form. The same lesson is to be gleaned from experiments in the case of the *hemisphere*, *cone*, and *circular plate* (all in base presentation), whose resistance is found to be approximately equal (Fig. 17).

§ 22. Wake and Counterwake Currents.—Reference has already been made to the *frictional wake current* to which a streamline body gives rise owing to the viscous stress it exerts on the fluid in its neighbourhood. With bodies of imperfect form there is, in addition to the frictional wake, a wake current constituted by the contents of the dead-water region, that is, the fluid contained within the surface of discontinuity.

The general motion of the wake current is in the same direction as the body itself, but, owing to the viscous drag exerted on it by the surrounding stream, this motion has superposed on it one of circulation, which probably results in the central portion of the wake travelling actually faster than the body[1] and the outer part slower, though Dines' experiments seem to point to the disturbance being of so complex a character that it is impossible to trace any clearly defined system.[2]

Now, since there can be no momentum communicated to the fluid in sum (§ 5), there must be surrounding the dead-water or wake current a counter-current in the opposite direction to that of the wake, that is, in the reverse direction to the motion of the body; and this *counterwake* current is being continuously generated, just as the wake current itself, and contains momentum equal and opposite to that of the wake. When in a fluid possessing viscosity the wake and counterwake currents intermingle by virtue of the viscous connection between them, and become involved in a general turbulence, the plus and minus momenta mutually cancel, and the final condition of the fluid at all points is one of zero momentum.

We may regard the counterwake current as a survival of the motion which, we have shown, must exist in the neighbourhood of the maximum section of a streamline body (§ 13) opposite in direction to its motion through the fluid. The failure of the stream to *close in* behind the body means that this motion will persist.

[1] Since writing this passage the author has observed this "overtaking" current photographed in Fig. 9. It may be faintly discerned in this Figure in the central region of the "dead-water."

[2] "On Wind Pressure upon an Inclined Surface," Proc. Royal Soc., 1890.

The mingling of the wake and counterwake may be regarded as a phenomenon quite apart from the initial disturbance, and the turbulence or otherwise of the wake does not materially add to or detract from the pressure on the front face of the body, but concerns merely the ultimate disposal of the energy left behind in the fluid.

No distinction is necessary between the frictional wake and the dead-water wake so far as the production of a counterwake current is concerned. The total wake current is the sum of the two, and the total counterwake is equal and opposite to the total wake.

§ 23. Streamline Motion in the Light of the Theory of Discontinuity.—The theory of kinetic discontinuity presents the subject of streamline motion in a new light, and enables us to formulate a true definition of streamline form. Thus—

A streamline body is one that in its motion through a fluid does not give rise to a surface of discontinuity.

In the previous discussion, § 9 *et seq.*, no attempt has been made to delineate streamline form, that is to say (according to the present definition), the form of body that in its motion through a fluid will not give rise to discontinuity. It has been assumed that such a body is a possibility, and from the physical requirements of the case the general character of the body form has been taken for granted.

Under our definition, if, as in the mathematical (Eulerian) theory, we assume *continuity* as hypothesis, then all bodies must be streamline, which is the well-known consequence. If, on the other hand, as in the Newtonian medium, *we assume discontinuity*, then it is evident by our definition that streamline form can have no existence, which, again, is what we know to be the case. It remains for us to demonstrate, on the assumption of the properties of an ordinary fluid, the conditions which govern the existence or otherwise of discontinuity, and so control the form of a streamline body.

In order that streamline motion should be possible such motion must be a stable state, so that, if we suppose that by some means a surface of discontinuity be initiated, the conditions must be such that the form of motion so produced is unstable.

Let us suppose that we have (Fig. 10) a streamline body made in two halves, and that the rear half, or *run*, be temporarily removed; then a surface of discontinuity will be developed, as indicated in the figure. Let now the detached portion be replaced. Then the question arises, What are the changed conditions that will interfere with the permanence of the discontinuous system of flow, as depicted in the figure?

If, in the first place, the fluid be taken as inviscid, and if,

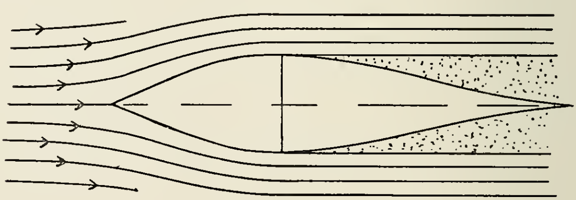

FIG. 10.

for the purpose of argument, we assume that the system of flow indicated in the figure is possible in an inviscid fluid, then it is evident that when the run is replaced we shall not have disturbed the conditions of flow, for our operations have been confined to the dead water region, where the fluid is at rest relatively to the body. Consequently the discontinuous system of flow will persist. That is to say, *under the supposed conditions* streamline motion is either unstable or is at best a condition of neutral equilibrium. Let us next introduce viscosity as a factor. The conditions are now altered, for the fluid in the dead-water region is no longer motionless, but is in active circulation, and the introduction of the rear half of the body obstructs the free path of the fluid, so that, as the outer layers of the dead-water are carried away by the viscous drag, the fluid in the interior has difficulty in finding its way back to take its place. This difficulty is greatest in the region from which the discontinuity springs, where the dead-water runs off to a "feather edge," and it is evident that some point of attenuation is reached at which the return flow becomes impossible, and the fluid will be "pumped out" or ejected from the region forward of this point. This brings the discontinuity further aft on the body, where the process can be supposed repeated, so that eventually the whole dead-water has been pumped away, and streamline motion supervenes. It is evident that the process will not occur in stages, as above suggested, but will be continuous.

It might be supposed from the foregoing argument that the degree of curvature of the surface of the body would not be a matter of importance, as in any case the *feather edge* of the dead-water would be sufficiently fine to ensure the ejection of some small amount of the fluid, and this process by continuous repetition would eventually clear the wake of its contents. If the surface of the body were *frictionless*, doubtless this might be the case, but it is established that there is *continuity* between the surface of an immersed body and the surrounding fluid; that is to say, there is the same degree of viscous connection between the fluid and the surface as there is between one layer of the fluid and another. The consequence of this is that the dead-water never fines off entirely, but extends forward as a sort of sheath enveloping the whole surface of the body, and if the curvature at any point is too rapid, the ejection may not prove effective, and the discontinuity will persist. It is evident therefore that there will be some relation between the bluffness of form permissible and the viscosity of the fluid, and, other things being equal, the less the viscosity the finer will have to be the lines of the body. The theory evidently also points to the importance of *smoothness of surface* when the critical conditions are approached.

The subject is not yet exhausted. We know that the thickness of the stratum of fluid infected by skin friction increases with the distance from the "cut-water"; that is to say, the factor on which the curvature of the surface probably depends is relatively more important on the buttock than on the shoulder. Hence we may expect that the lines of entrance can with impunity be made less fine than the lines of the run.

Again, all forces due to the inertia of the fluid vary as the

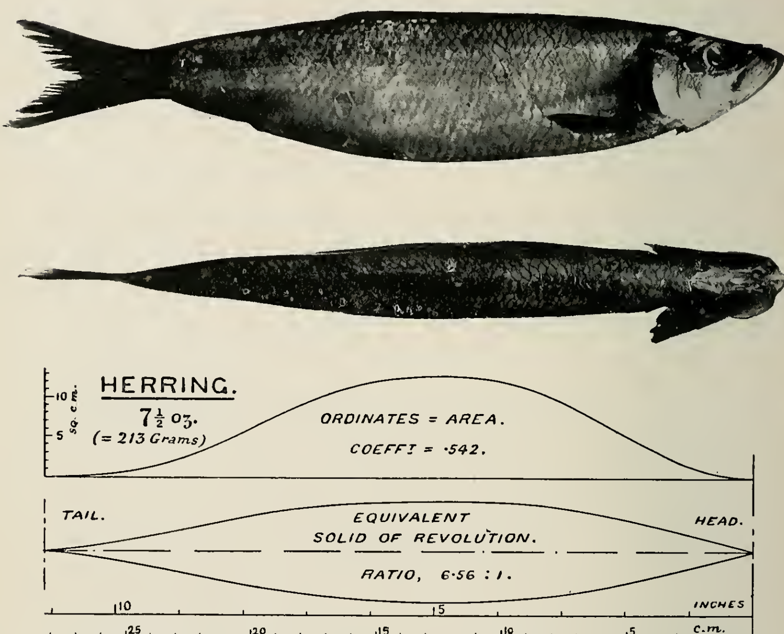

Fig. 11.

square of the velocity; those due to viscosity vary in the direct ratio of the velocity (§ 31). Therefore for different velocities the influence of viscosity predominates for low velocities, and that of inertia when the velocity is high. Consequently the form suited to high velocity will be that appropriate to low viscosity, and *vice versâ;* that is to say, the higher the velocity the finer will be the *lines* required.

§ 24. Streamline Form in Practice.—The practical aspect of streamline form may be best studied from the bodies of fishes and birds, the lines of which have been gradually evolved by nature to meet the requirements of least resistance for motion through a fluid, water or air, as the case may be.

Since all animals have functions to perform other than mere

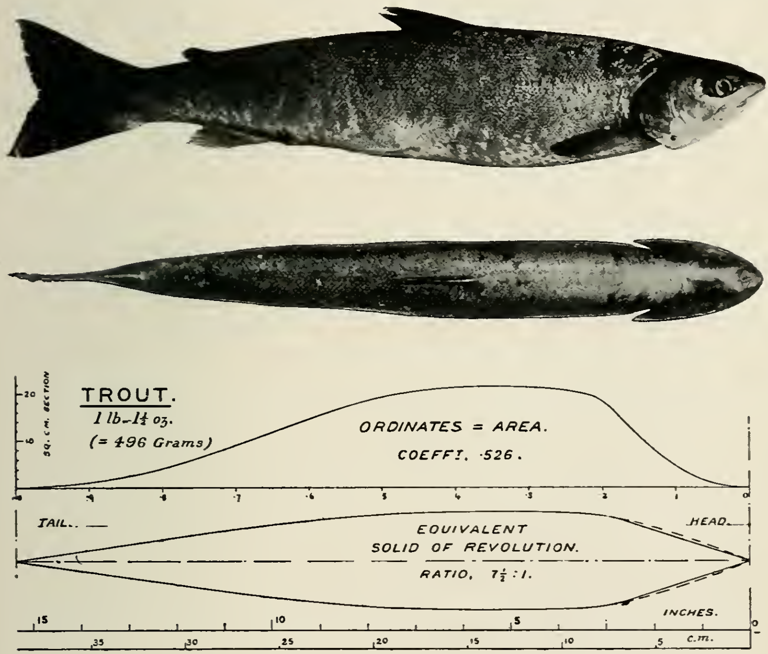

FIG. 12.

locomotion, we find great diversity of detail, and we frequently meet with features whose existence is in no way connected with the present subject. We may readily recognise in these cases the exceptional development of certain organs or parts to meet the special requirements of a particular species, and by a sufficiently wide selection we can eliminate features that are not common, and so arrive at an appreciation of that which is essential. Thus the *herring* (Fig. 11), the *trout* (Fig. 12), or

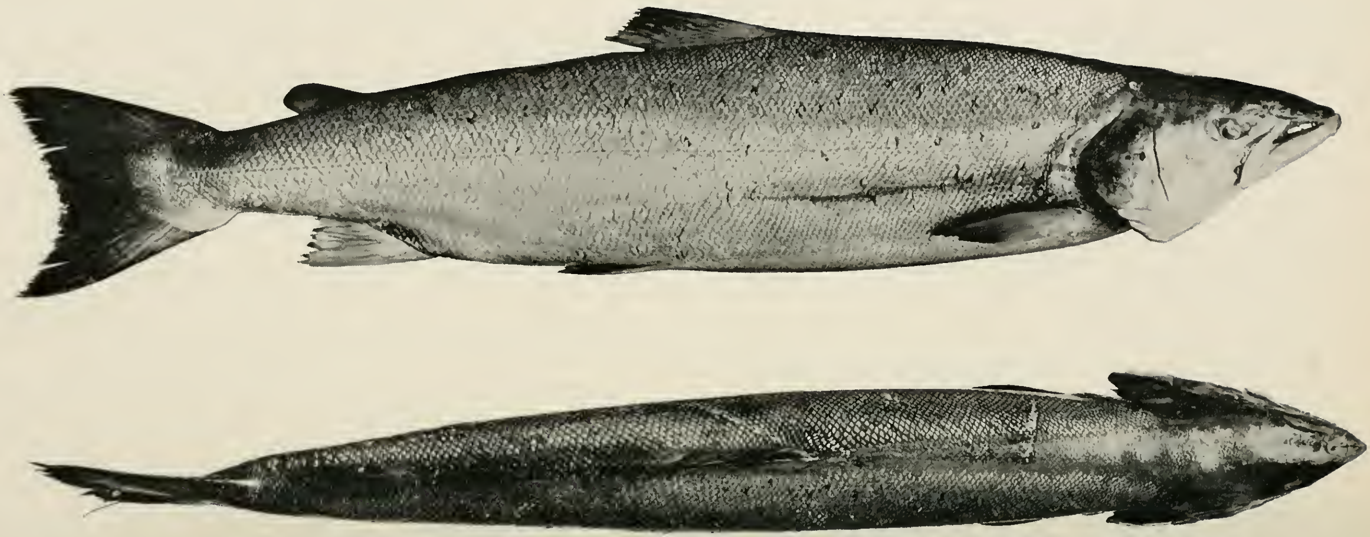

Fig. 13.

the *salmon* (Fig. 13) may be cited as typically *fish-shaped fish*.

Beyond the lessons to be derived from these natural forms, there is very little practical information available. The lines of ships are governed by considerations foreign to the subject, the question of wave-making, for example, being a matter of vital importance. The *submarine* has not yet reached a stage of development that would justify its form being taken as a fully

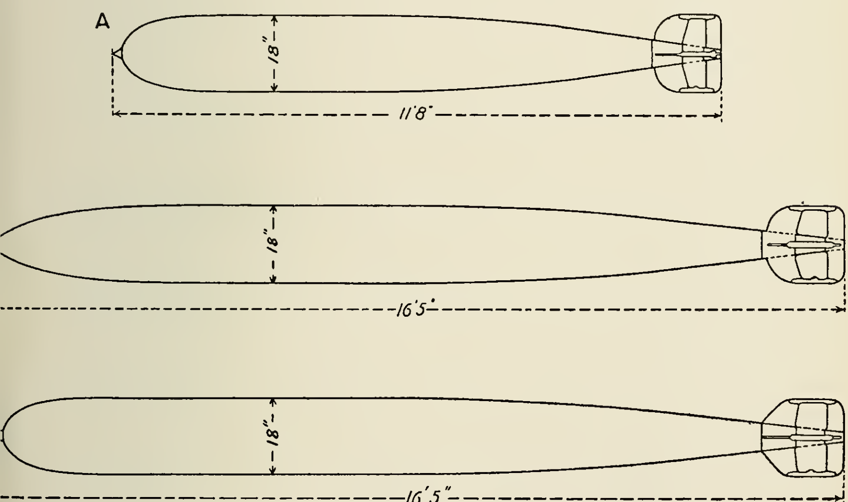

Fig. 14.

evolved model; also, for obvious reasons, this type of vessel is one of which but little information has been published.

In Figs. 11 and 12 curves are given whose ordinates represent the area of cross-section at different points. This curve has been obtained by *differentiating* a displacement curve plotted from a series of *immersion measurements*. These measurements were made by a method of displacement, the fish, suspended tail downward, being lowered stage by stage into a vessel of water, measurements being made of the overflow.

The area curves have been further translated into the form of *solids of revolution*, which may be taken as the equivalent of the original form in each case. Some doubt exists as to the exact form in the region of the head, owing to the water entering the gills. The effect of this is very evident in the case of the trout (Fig. 12), where the form has been "made good" by a dotted line.

For the purpose of comparison outline elevations are given in Fig. 14 of three types of Whitehead torpedo. These are forms that have been developed by long experience, but the shape is largely dictated by special considerations. The bluff form of head, for example, in models *A* and *C* is adopted in order to bring the explosive charge into as close proximity as possible to the object attacked. It probably also gives a form that is more easily steered.

§ 25. Streamline Form.—Theory and Practice Compared.—Before a rigid comparison can be instituted between the theoretical results of § 23 and the actual forms found in nature considerable further information is required. We do not know with accuracy the speeds for which the different fish forms have been designed or are best adapted. We also lack knowledge on certain other important points. The present comparison must therefore be confined to generalities.

In the first place, we may take it that the conclusion as to the bluffer form being that suited to greater viscosity is fully borne out in practice, though the whole of the considerations bearing on this point are not here available. It is explained in Chap. II. that the viscosity *divided by density* (or *kinematic viscosity*) is the proper criterion in such a case as that under discussion, and on this basis air is far more viscous than water, so that we shall expect to find aerial forms bluffer in their lines than aqueous forms. Taking the solid of revolution as the basis of comparison, we have in the case of the herring and the trout the length approximately seven times the maximum diameter. The general ratio found amongst bird forms is about three or four to one, the samples chosen for measurement being as far apart as the albatros and the common sparrow. Consequently we find that the theoretical conclusion receives substantial confirmation.

The relation of fineness to speed is not so easy of demonstration, owing to the absence of accurate data. It would, however, seem to be sufficiently obvious as a matter of general experience that our conclusions hold good. It is almost certain that in general the fish with the finer lines are the faster swimmers. If this conclusion be accepted, the viscosity relation of the preceding paragraph is emphasised, for there is no doubt that the average speed of flight is greatly in excess of any ordinary velocity attained by fish.

§ 26. Mutilation of the Streamline Form.—There are certain types of body that may be regarded as *mutilations* of the streamline form, and the consequences of such mutilation may now be examined.

If, in the case of a body propelled at a constant velocity, the entire run be removed, as in § 23, the consequence is a surface of discontinuity emanating from the periphery of section. Under these circumstances, if we neglect the influence of viscosity and the consequent loss of wake pressure, the work done appears wholly in the counterwake current, on the production of which energy is being continuously expended. This performance of work is otherwise represented by a resistance to motion, being the difference between the excess pressure on the head and the diminished pressure on the shoulder, according to the principle explained in § 11. If now we restore the buttock, so that the mutilation is confined to the simple loss of the tail (Fig. 15), the diminished pressure on the buttock acts as a drag upon the body, and more work must be expended in propulsion. This additional energy will appear in the fluid as a *radial component* in the motion of the stream which does not exist if the whole *run* is removed. It is probable that some of this energy is restored by an increase in the pressure of the dead water due to the converging stream, but we have no means of making a quantitative computation.

An illustration of this principle may be cited in the type of hull employed in a modern racing launch. The stern is cut off square and clean, and may constitute the maximum immersed section. There would seem in fact to be no logical compromise between a boat with an ordinary well-proportioned *entrance* and *run*, and one in which the latter is sacrificed entirely. In such a form, when travelling at high speed the water quits the transom entirely, and consequently sacrifice is made of the hydrostatic pressure on the immersed transom area. The point at which

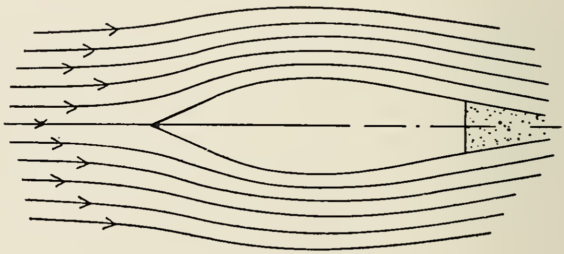

Fig. 15.

the *front half* of a boat thus takes less power for its propulsion than the whole is probably about that speed at which the skin friction on the *run* (the after-half), if present, exceeds the hydrostatic pressure on the maximum immersed section. This does not, however, determine the point at which it pays to make the sacrifice, owing to the fact that for the *same capacity* the truncated form has to be that of a larger model. The *rating rule* also exerts an arbitrary influence. When, as is usual, the length is penalised, an additional inducement is offered for the designer to adopt the truncated type. When the truncated type of hull is adopted it is advantageous to employ *shallow draught*, for the hydrostatic pressure for a given displacement is less. This form is also partly dictated by considerations relating to propulsion.

§ 27. Mutilation of the Streamline Form (continued).—In Fig. 16, *A* and *B*, the consequences of truncating the fore body, or *entrance*, of a streamline body are indicated diagrammatically. If, as in *A*, the mutilation be slight, the result may be merely a local disturbance of the lines of flow. A surface of discontinuity will probably arise, originating and terminating on the surface of the body in the manner shown. It is possible that if the streamline body be travelling at something approaching its critical velocity (at which even in its complete form it is on the point of giving

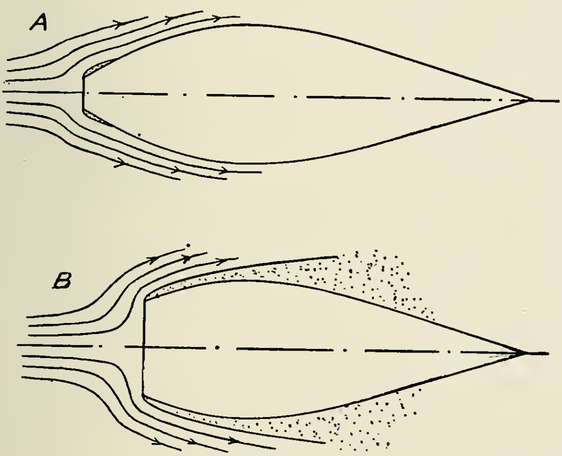

FIG. 16.

rise to discontinuity), a minor mutilation such as here suggested might have more serious consequences.

If the greater part of the *entrance* be removed, as shown at *B*, the surface of discontinuity generated quits the body for good, and the resistance becomes immediately as great as that of a normal plane of area and form equal to that of the section. This is in harmony with the experiments of Hutton and Dines, to which reference has already been made (Fig. 17), the three bodies shown being found to offer the same resistance within the limits of experimental error.

It is evident that the dictum of the late Mr. Froude, that it is "blunt tails rather than blunt noses that cause eddies" (and therefore involve a loss of power), is applicable only to bodies having already some approximation to streamline form. It is obviously useless to provide a nice sharp tail if previous attention has not been given to the shoulder and buttock lines. Mr. Froude probably meant that in a well-designed streamline form the tail should be finer in form than the head, a matter that up to his time had presumably been neglected.

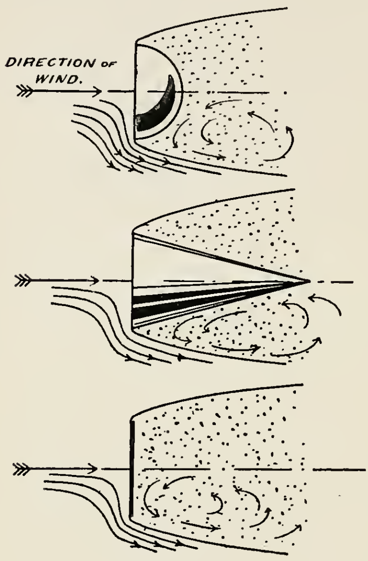

FIG. 17.

The primary importance of easy shoulder lines has been long recognised as a fundamental feature in the design of projectiles. A full-sized section of a Metford ·303 bullet, illustrating this point, is given in Fig. 18, and a streamline form of which it may be regarded as a "mutilation" is indicated by the dotted line.

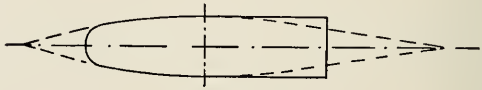
FIG. 18.

§ 28. **Streamline Flow General.**—Let us suppose an approximate streamline form to be built of bricks, and, in the first place, we will assume that the bricks are so small as to merely give rise to a superficial roughness. Then this roughness will add to the skin friction and will give rise to some local turbulence, but the general character of the flow system remains as before. We may go further and suppose the bricks so large as to form steps capable of giving rise to surfaces of discontinuity (Fig. 19). Then the resistance will be increased, and the layer of fluid next the body will be violently stirred up; but if we examine the fluid some distance away we shall still find it comparatively unaffected. If we now suppose the body to consist of a few large blocks, the depth of fluid affected by turbulence will be greater, but at a sufficient distance away we may still expect to find lines of flow of characteristic streamline form. We may therefore generalise and say, *All bodies passing through a fluid are surrounded by a streamline system of flow of a greater or less degree of perfection depending upon the conformability or otherwise of the surface or surfaces of the body.*

This proposition, if not sufficiently obvious from the

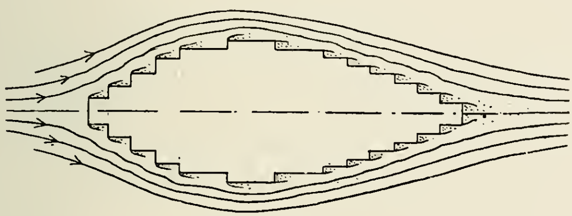

FIG. 19.

considerations above given, may easily be demonstrated experimentally.

In the experiment described in § 17, the orbital motion of the particles of the fluid is demonstrated by the motion of an ichthyoid body in air irregularly charged with smoke. This orbital motion, with its consequent displacement, is quite characteristic, and if other shapes of body be substituted for the streamline form, the motion of the fluid a short distance away is not sensibly affected. In the case of a body of streamline form, the motion can be observed much closer to the axis of flight than is the case for a sphere or other bluff form; also when the movement is complete nothing further happens. In the case of a sphere, the looked-for movement duly takes place; but immediately after the whole of the fluid under observation is involved in a state of seething turbulence, where the wake and counterwake currents are mingling. If the point of observation is sufficiently remote, the orbital motion may be detected, even in the case of the normal plane, beyond the immediate reach of the wake turbulence.

§ 29. Displacement due to Fluid in Motion.—It has been shown (§ 15) that the fluid in the neighbourhood of the path of flight of a streamline body undergoes displacement, and that the total displacement is equal to the volume of the body. It might be expected in the case of the normal plane, which possesses no volume, that the displacement would be *nil*, and such would doubtless be the case if the form of flow were that of the Eulerian theory.

In actuality the normal plane, in common with bodies of bluff form, carries a quantity of fluid bodily in its wake, which from the present point of view becomes in effect part of the body, so that the displacement manifests itself just as if the plane were possessed of volume. This is characteristic of all bodies that give rise to discontinuous motion; the displacement is greater than the actual volume of the body. If there were no mingling of the wake and counterwake currents, the displacement would be infinite, for the counterwake current would persist indefinitely.

In the case of a streamline body, a certain amount of fluid is carried along with the body by viscosity, and this similarly increases the effective displacement volume.

It would appear from actual observation that, where the displacement is due to the attendant fluid, the outer streamlines have a motion closely resembling that produced by a streamline body, but that those nearer the axis of flight terminate in the turbulent wake; the commencement of the orbit is all that can be seen.

§ 30. Examples illustrating Effects of Discontinuous Motion.—On the practical importance of the study of motion of the discontinuous type it is unnecessary to dwell. It is at present the only basis on which it is possible to account for the phenomenon of fluid resistance as experimentally known. Beyond this there are many examples and illustrations which are of especial interest, considered either as proofs of the theory itself or in relation to their actual consequence or utility.

A useful application of the principle is found in the screen employed on fast steamships to protect the navigating officer, and frequently the "watch," from the rush of air, without

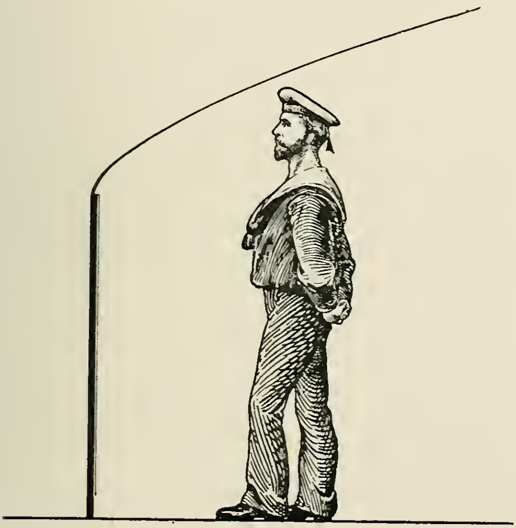

Fig. 20.

obstructing the field of vision. This is illustrated diagrammatically in Fig. 20, in which it will be seen that the live stream is carried clear over the sailor's head, the latter being protected by the *surface of discontinuity.* A similar device is frequently adopted in connection with the dashboard of a motor car.

Evidence of the most striking kind of the existence of a surface of discontinuity is sometimes met with in the growth of trees in the immediate vicinity of the edge of a cliff (Fig. 21). It may be seen that the form of the surface is clearly delineated, the tree top being cut away as though it might have been sheared off by a stroke of a mighty scythe.

An interesting example of an indirect effect of discontinuity is to be found in the effect of "cut" or "side" on the flight of a ball. Let a ball (Fig. 22) moving in the direction of the arrow *A* have a spin in the direction of the arrow *B*. Now where the

Fig. 21.

direction of motion of the surface of the ball is the same as the relative motion of the fluid, as at *D*, the surface will assist the stream in ejecting the dead water, so that the discontinuity will be delayed, and will only make its appearance at a point some distance further aft than usual. On the other hand, on the side that is opposing the stream the surface of the ball will pump air in, and so assist the discontinuity, which will make its appearance prematurely. The net result of this is that the counterwake will have a lateral component (downwards in the figure), and, on the principle of the continuous communication of momentum, there will be a reaction on the ball in the opposite direction, that is to say *upwards*. A ball may therefore be sustained against gravity or be made to "soar" by receiving a spin in the direction shown, or, if the spin be about a vertical axis, the path of the ball will be a curve (in plan), such that the aerodynamic reaction will be balanced by centrifugal force.

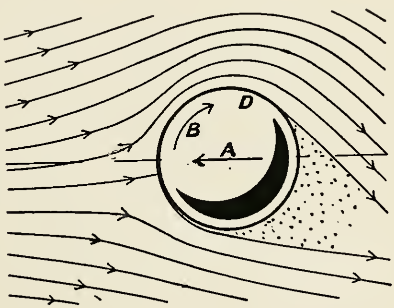

FIG. 22.

The actual *means* by which the reaction acting on the ball comes about may be understood from either of two points of view. We may (Fig. 22) regard this reaction as the centrifugal effect of the air passing *over* the ball preponderating greatly over that of the fluid passing *underneath*, or if we anticipate a knowledge of hydrodynamic theory (Chap. III.), we know that the greater proximity of the lines of flow in the former region is alone sufficient to indicate diminished pressure. The lines as drawn in the figure are not plottings—there is no way known of plotting a field of flow of this degree of complexity—but they may be taken as a very fair representation of what the plotting would be if it could be effected.

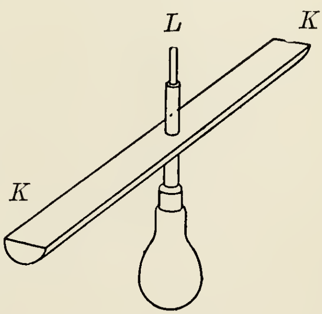

FIG. 23.

The reason that the streamlines have been shown rising to meet the ball in its progress will be better understood in the light of Chaps. III. and IV. This detail is related to more advanced considerations than can be entered into at present.

A further interesting example is found in the *aerial tourbillion*[1] (Fig. 23), in which the rotor K is a stick of segmental section mounted to revolve freely about the axis L. The plane face of the rotor is set truly at right angles to the axis of rotation. If this apparatus be held in a current of air with the plane face fronting the wind, as, for instance, by holding it outside the window of a railway carriage in motion, the rotor evinces no tendency to go round in the one direction or the other. If, however, a *considerable* initial spin be imparted in either direction, the wind will suddenly *get a bite*, so to speak, and the rotor will gather speed

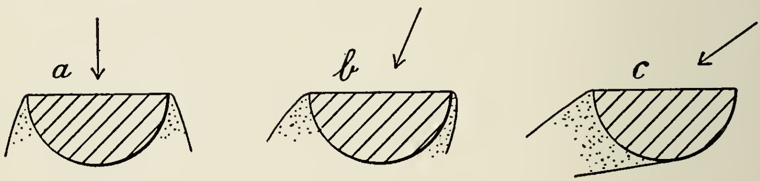

Fig. 24.

and spin at an enormous rate, as if it were furnished with sails like a well-designed windmill.

Referring to Fig. 24, we have at *a* the type of flow illustrated to which the blade of the rotor will give rise when its motion is normal to the air; *b* similarly indicates the form of flow when the rotor is going round slowly, not fast enough for the air to *take hold*. In both these figures we have the flow independent of the "rear body form," and the rotor behaves just as if it were a flat plate. Now, let us suppose that the rotor be given a sufficient initial spin to bring about the state of things represented at *c*.

[1] This interesting aerodynamic puzzle was first brought to the notice of the author by Mr. Henry Lea, consulting engineer, of Birmingham, who, it would appear, had it communicated to him by Mr. A. S. Dixon, who in turn had it shown him when travelling in Italy by Mr. Patrick Alexander. The author has taken no steps to trace the matter further. The explanation here given is his own.

The surface of discontinuity that ordinarily springs from the leading edge has got so close to the rear body of the rotor as to have *ejected the "dead-water"* on that side, and the resulting form of flow will be something like that illustrated in Fig. 25. Here the pressure on the left-hand side (as shown) will be that of the "dead-water," which is, as we know, somewhat less than that of hydrostatic head, while that on the right-hand side will, owing to the centrifugal component of the stream, be very much lower; that is to say, the rotor will experience a force acting from left to right which is in the direction of the initial spin, so that the motion will be accelerated and will continue. The fact that the propelling force only comes into existence when the initial spin is sufficient to eject the dead water from the leading side of the rotor blade fully explains the observed fact that a very considerable initial spin is necessary.

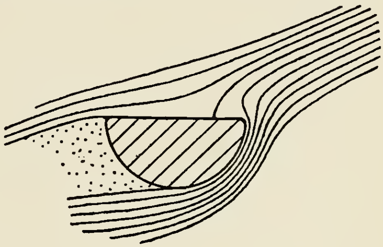

FIG. 25.

CHAPTER II.

VISCOSITY AND SKIN-FRICTION.

§ 31. **Viscosity.—Definition.**—The fundamental law of viscosity is enunciated in the form of an *hypothesis* to Section IX., Book II., of Newton's "Principia," as follows :—

The resistance arising from the want of lubricity in the parts of a fluid is, cæteris paribus, *proportional to the velocity with which the parts of the fluid are separated from each other.*

The subsequent propositions li., lii., and liii., show that the expression "want of lubricity" is synonymous with the modern term "viscosity," and the motion contemplated by Newton in framing the foregoing hypothesis is motion in *shear*. The Newtonian law has since received ample verification at the hands of Maxwell and others.

Maxwell, in his "Theory of Heat," gives a quantitative definition of viscosity as follows :—

The viscosity of a substance is measured by the tangential force on the unit area of either of two horizontal planes at a unit distance apart one of which is fixed while the other moves with the unit of velocity, the space between being filled with the viscous substance.

Or if (Fig. 26) a stratum of the substance of thickness l be contained between a fixed plane $A\ B$ and the plane $C\ D$, moving from C towards D with a velocity V, then, when a steady state is established, the motion of the intervening fluid will be in the direction C to D, and its velocity at different points will be in proportion to the height above the plane $A\ B$, so that the fluid in immediate contact with the plane $A\ B$ will remain at rest, and that in immediate contact with the plane $C\ D$ will have the velocity V in common with it. Then, if F be the horizontal force applied to the plane *C D per unit area* to overcome the resistance of the fluid, we have—

$F = \mu \frac{V}{l}$, where μ is a quantity termed the *coefficient of viscosity.*

This equation is merely the algebraic expression of the law previously stated, for where V and l are unity we have $F = \mu$.

It will be seen that between the planes *A B* and *C D* there will exist a *velocity gradient.* A series of particles situated at points on a straight line *a, a, a, a,* at one instant of time, will be situated at points *b, b, b, b,* on another straight line at another instant,

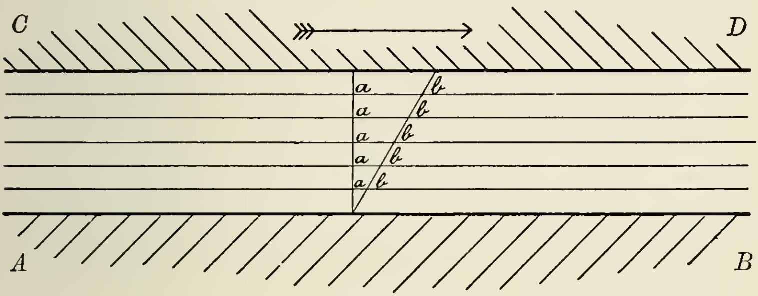

Fig. 26.

the figure thus giving a pictorial idea of the motion in a viscous fluid.

§ 32. Viscosity in relation to Shear.—In the foregoing illustration, which is in substance as given by Maxwell, the nature of viscous strain as a *shear* is sufficiently obvious. There are cases, however, in which viscosity plays a part in which the conditions are not so straightforward. The modern definition of shearing stress is *stress that tends to alter the form of a body without tending to alter its volume,* and any strain that involves the *geometrical form or proportions* of a body requires shearing stress for its production. All stresses and strains can be resolved into *shear* and *dilatation* (plus or minus); and such stresses as linear tension or compression of a solid involve *stress in shear.*

We thus see that changes in the shape of a body of a fluid, such as take place in the course of its passage through a "tube of flow" in the vicinity of a streamline body, are resisted by viscosity in proportion to the velocity with which the change of form takes place, and the work done on the fluid in this manner must be supplied by a propulsive force; that is to say, the body will be resisted in its motion through the fluid from the cause stated. We have here *one* of the causes of viscous resistance.

We cannot state that this form of resistance will increase directly as the velocity, that is, according to the viscous law, for we do not know that the form of the lines of flow is the same at different velocities. It would appear that this must be so for an inviscid fluid. It would also seem evident that the viscous resistance will modify the form of flow materially. It may therefore be deduced that the form of flow will be more modified for low than for high velocities, in which case the form of resistance we are now discussing will not vary exactly in the direct ratio of the velocity.

Bodies other than of streamline form will also be affected by this type of viscous resistance, when it will appear as an added resistance. The only exception is found in the case of a *plane moving tangentially*, the consideration of which introduces the important subject of *skin-friction*.

§ 33. Skin-friction.—It is well established that there is no slipping of a fluid past the surface of a solid, but that the film adjacent to the surface adheres to it, and the resistance experienced is of the nature of a viscous drag. This fact has already been assumed in the discussion of the law of viscosity, for otherwise there would be no necessity for the fluid to be set in motion by the plane $C\,D$ at all. To a certain extent, therefore, the term "skin-friction" is misleading. It is, however, a term sanctioned by usage, and it is difficult to find a more suitable expression.

Let us suppose that a plane having no sensible thickness be put in motion tangentially through a fluid, and be maintained in motion until a steady state is reached. Then the advancing edge of the plane is continually engaging with new masses of the fluid, and setting them in motion by virtue of the viscous stress exerted. But the conditions under which any given mass of fluid is acted on are not those of the previous hypothesis; the force resisting the motion of the plane is that of the *inertia of the fluid itself*; and if we confine our attention to any one portion of the fluid, its condition is not that of a steady state, but one of acceleration. Now it is evident that when the leading edge first enters the undisturbed region the stratum of fluid affected will be quite thin; and as the following portions of the plane successively traverse the same region the thickness of the stratum set in

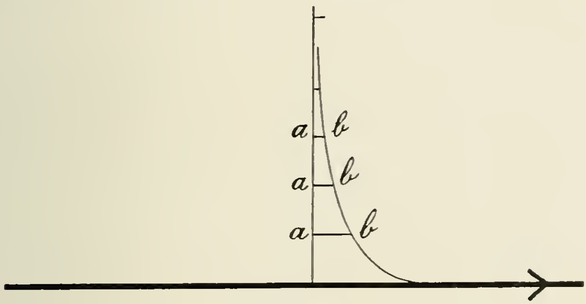

FIG. 27.

motion continuously increases, and the *velocity gradient* will correspondingly diminish. This is illustrated in Fig. 27, in which the line *a*, *a*, *a*, represents the original position of a series of particles of the fluid at some given instant, and *b*, *b*, *b*, the position assumed after a short time has elapsed.

§ 34. Skin-friction.—Basis of Investigation.—Owing to the considerations dealt with in the preceding section, it is evident that we cannot regard skin-friction as of necessity amenable to the ordinary viscous law, *i.e.*, $F \propto V$ where l and μ are constant; it is in fact easy to prove that this law will not apply.

Let us suppose for example that the motion of the fluid is *strictly homomorphous* in respect of changes of V, that is to say, if u, v, w, be the velocity of the fluid particles at any instant of time in the direction of co-ordinates x, y, z, travelling with the body (or plane), we are supposing that any variation in V results in a variation in like ratio of u, v, w, for all values of x, y, z.

In order to simplify the thinking in connection with this problem it is convenient to suppose the body to be a plane travelling in the direction of the axis of x, and confine our attention to the motion of the fluid taking place in like direction. Let y be taken as the axis at right angles to the plane.

The viscous stress at every point will be proportional to the velocity gradient, that is $\frac{du}{dy}$, which on the present supposition varies as V for every point x, y, z, in the region, consequently we shall have $F \propto V$, which is the viscous law. Now if the prescribed conditions satisfy the dynamic requirements of the problem, we might conclude that the motion is strictly homomorphous, and that the viscous law obtains, but such is not the case. The momentum communicated per second to any given layer of the fluid, and therefore to the whole fluid, is = mass $\times$ the velocity per second imparted, that is $\propto u\, V \propto V^2$; so that under strictly homomorphous conditions the viscous stress cannot be satisfied for varying speeds by the inertia of the fluid.

If, when the velocity V increases, we suppose that the layer of fluid affected to any given degree becomes thinner, and *vice versa*, it is clear that the viscous forces will rise in a greater ratio than directly as V, for the velocity gradient will be steeper, also the inertia forces will be less, for the mass of fluid to be set in motion will be less. It is, therefore, evident that we may suppose the thickness of the affected strata varies with the velocity in the degree necessary to preserve a balance between the *viscous* and *inertia* forces.

§ 35. **Law of Skin-friction.**—Let us suppose that in any two systems, differing only as to velocity V, the whole region be divided into strata by an imaginary series of equidistant planes, so that the thickness of corresponding strata in the two systems, and their distance from the *material* plane, shall be in the constant ratio n. And let us denote the distance between adjacent planes by the symbol $\triangle y$, and the corresponding velocity difference in the axis of x by $\triangle u$. Then $\triangle y \propto n$ and $\triangle u \propto V$. (1)

Let F = the tangential force.

We have F *as measured by viscosity* varies as the area (which is constant) $\times$ velocity gradient, or—

$$F \propto \frac{du}{dy} \propto \frac{\triangle u}{\triangle y} \propto \frac{V}{n}. \qquad (2)$$

And F *as dependent on dynamic considerations* = momentum imparted per second to the fluid. For unit width of any stratum we have mass $= \rho \triangle y \ V$, and velocity varies as V or $F = \Sigma \rho \triangle y \ V^2$.

$$\therefore \qquad F \propto nV^2. \qquad (3)$$

By (2) and (3) we have—$nV^2 \propto \frac{V}{n}$,

or $$n \propto V^{-\frac{1}{2}},$$

substituting in either (2) or (3)—

$$F \propto V^{1.5}. \qquad (4)$$

This may be taken as the *normal law of skin-friction.*[1]

§ 36. **Kinematical Relations.**—In dealing with problems relating to fluid resistance it is found to lead to simplification to eliminate the density of the fluid by introducing two new quantities, *kinematic resistance* and *kinematic viscosity.*

Kinematic resistance, which we will denote by the symbol R, may be defined as the resistance per unit density, or $R = F/\rho$, and is consequently of the dimensions $\frac{L^4}{T^2}$.

[1] The foregoing demonstration is here presented for the first time by the author; the experimental fact was discovered by Mr. H. S. Allen (compare § 50). The relation $F/V^{1.5}$ = const. may appropriately be termed *Allen's law.* It is evident in the above investigation that the balance of viscous and dynamic forces is demonstrated for all corresponding layers of the region each to each, for any number of cases of V variation, and consequently the method is comprehensive, and includes both the plus and minus momentum of the wake and counterwake currents.

Kinematic viscosity, which we will denote by the symbol $\nu = \mu/\rho$, and is consequently of the dimensions $\frac{L^2}{T}$.

Writing the law of viscous resistance in its kinematic form we have $R = A\,\nu\,\frac{V}{l}$ where A is the area of the surface; it will be noted that this expression is dimensional.

If we similarly write the law of skin-friction $R = A\,\nu\,\frac{V^{1.5}}{l}$, we find that the dimensions do not harmonise.

Let us examine this expression in a general form, where

$$R = c \times A^p \times \nu^q \times V^r;$$

dimensionally:—

$$\frac{L^4}{T^2} = c \times L^{2p} \times \frac{L^{2q}}{T^q} \times \frac{L^r}{T^r};$$

$$\therefore \quad 2p + 2q + r = 4,$$

and

$$q + r = 2,$$

$$\therefore \quad q = 2 - r,$$

$$\therefore \quad 2p + 4 - 2r + r = 4,$$

$$\therefore \quad 2p - r = 0,$$

$$p = \frac{r}{2}.$$

The general expression therefore becomes :—

$$R = c\,\nu^q\,A^{\frac{r}{2}}\,V^r \qquad (5)$$

in which

$$q + r = 2.$$

This is the general equation to the kinematic resistance of bodies in viscous fluids, and correlates the variations in respect of viscosity, area, and velocity; the application extends to both normal and inclined planes and bodies of the most diverse form.[1] It may be illustrated here in its relation to the law of skin-friction; we have, R varies as $V^{1.5}$ and the full kinematic expression therefore becomes—

$$R = c\,\nu^{\frac{1}{2}}\,A^{\cdot 75}\,V^{1.5}, \qquad (6)$$

[1] The method of dimensions presumes geometrical similarity of figure.

and we have the unexpected but experimentally established result that the resistance does not vary with the area, but according to a fractional power of same.

If, as is customarily assumed, the resistance of a body is taken as proportional to the square of the velocity, then we shall have q = zero, and the pressure is independent of viscosity altogether; this result is due to Allen.[1] Under these conditions the resistance is directly as the area, and conversely if viscosity have any influence on the resistance, then the resistance cannot vary directly as the area, hence the existence of viscosity may be regarded as giving a definite *scale* to the fluid.

§ 37. **Turbulence.**—The steady state of viscous motion depicted in Figs. 26 and 27, on which the laws of viscosity and skin-friction have been based, is found in practice to obtain over a moderate range of velocity only. When a certain critical velocity is exceeded the continuity becomes broken and the phenomenon of *turbulence* manifests itself. Under conditions involving pure viscosity (in contradistinction to the more complex phenomenon of skin-friction), this critical point has been investigated experimentally by Mr. Osborne Reynolds in the case of liquid flowing through a straight tube. It is found that up to a certain velocity the flow is everywhere parallel to the axis, but when this critical velocity is reached the parallel flow breaks up, and is replaced by an irregular turbulent motion. Up to the critical velocity the law deduced by Poissuille for viscous flow through a tube holds good; beyond this point the resistance rises more rapidly, and for high velocities approximates to F varies as V^2, when the energy is mostly expended in generating the turbulent motion.

The method of investigation employed by Osborne Reynolds consisted of observing the behaviour of a coloured filament of liquid introduced in the centre of a tube containing liquid in motion; the result obtained is that steady motion ceases to exist

[1] *Phil. Mag.*, September and November, 1900.

if the mean velocity exceeds $\frac{1000\,\nu}{a}$ where a is the radius of the tube[1] (c.g.s. units).

§ 38. General Expression.—Homomorphous Motion. — Let us examine generally the relations of *geometrically similar systems* possessed of *homomorphous motion*—that is, under circumstances when the theory of dimensions is strictly applicable, then the quantities upon which the motion depends are comprised by—velocity = V, kinematic viscosity ν, and a linear (scale) dimension l.

Let us write

$$l = c\, V^p\, \nu^q,$$

or, in terms of dimensions

$$\mathrm{L} = \frac{\mathrm{L}^p}{\mathrm{T}^p} \times \frac{\mathrm{L}^{2q}}{\mathrm{T}^q},$$

and we have the equations

$$p + q = 0,$$
$$p + 2\,q = 1,$$
$$\therefore \quad q = 1 \qquad p = -1,$$
$$\therefore \quad l = c\,\frac{\nu}{V} \quad \text{or} \quad V = c\,\frac{\nu}{l}, \tag{7}$$

which may be taken as the general equation connecting all similar systems of flow in viscous fluids.

In a number of tubes, such as may be supposed employed for experimentally investigating the phenomenon of turbulence, we have a number of such similar systems, and it will be noted that the expression is identical with that arrived at by Mr. Osborne Reynolds, in whose equation we have $c = 1000$ as expressing a particular state of motion.

§ 39. Corresponding Speed.—The above expression enables us to formulate at once a *law of corresponding speed* for motion in any viscous fluid, for, if the physical properties of the fluid do not vary in any way ν will be constant and we have $V \propto \frac{1}{l}$, that

[1] Poynting and Thomson, "Properties of Matter," Chap. XVIII.

is to say, for *submerged* model experiments, in which the condition of *acceleration* = *constant* does not apply, the smaller the model the *higher the speed*, in the direct proportion of the linear dimension—a rather unexpected result.

The law of corresponding speeds employed in naval architecture is primarily influenced by considerations of wave-making, in which (as shown later in the present work) the dimensional basis is acceleration $\left(\frac{L}{T^2}\right)$ = constant; the author has proved from aerodonetic considerations that the same law obtains in connection with aerial flight.[1] In this law we have V varies as the square root of l so that the two laws are incompatible—that is to say, not capable of simultaneous fulfilment. This fact is well known in connection with model experiments relating to ship resistance, the results of experiment being subject to correction according to certain rules for frictional resistance, and similar correction will be required in the case of aerodrome experiments.

If it were possible, as by employing some different fluid, to alter the value of ν when experimenting with scale models, the necessity for applying a correction might be obviated; we have :—

By Froude's law $V^2 = c_1 l$, where c_1 is a constant.

By Equation (7) $V = c\frac{\nu}{l}$, or $\sqrt{c_1 l} = c\frac{\nu}{l}$, that is, $c_1^{\frac{1}{2}}\, l^{\frac{3}{2}} = c\,\nu$ (8)

or, the kinematic viscosity is required to vary with the 3/2 power of the linear dimension.

We cannot always obtain fluids with viscosity to order, but if we select two fluids such as air and water, whose kinematic viscosities are, at 15° C., in the approximate ratio of 14 : 1, and if l_1 and l_2 represent the lengths of the two models, and ν_1 and ν_2 the values of the viscosities respectively, then,—

$$\frac{l_1}{l_2} = 14^{\frac{2}{3}} = 5{\cdot}8.$$

That is to say, that a model aerodrome, made to a $\frac{1}{5{\cdot}8}$th scale

[1] Aerodonetics.

and adapted for motion under water, will, at a velocity proportioned to the square root of its linear dimension, that is $\frac{1}{2{\cdot}4}$th the full scale velocity, give rise to a geometrically similar disturbance in the fluid, and will itself undergo geometrically similar disturbance, and density for density the resistance will be proportional to the cube of the linear dimension—that is to say, in the ratio of $\frac{1}{196}$ of the full scale model; or, taking count of the relative density of air and water, the resistance of the smaller model will be approximately four times that of the greater.

§ 40. **Energy Relation.**—In all cases of purely viscous resistance the law of viscosity requires that the resistance shall vary directly as the velocity; and the whole of the energy expended disappears at once into the thermodynamic system. In cases where the resistance is *dynamic*—that is to say, where it is due to the continuous setting of new masses of the fluid in motion—the whole of the energy expended remains in the fluid in the kinetic form (being only subsequently frittered away), and the resistance varies as the square of the velocity. Where the resistance is due to both causes combined, as in the case of skin friction, the portions of the total resistance varying directly, and as the square, are respectively proportional to the energy expended in the two directions.

Now for any particular velocity, the total resistance—that is, the sum of the viscous and dynamic resistances—may be expressed as varying as the n th power of the velocity; it is not necessary that the value of n should be constant over the whole range of the R V curve; it may be a quantity varying as a function of V, the form of which is unknown; but, for the particular value of V chosen we have

$$\frac{dR}{dV} = n\ V^{(n-1)}, \text{ or } = n\ \frac{R}{V}.$$

Let R_1 be the resistance varying as V, and R_2 be the resistance varying as V^2, then $R = R_1 + R_2$, and we have

$$\frac{dR}{dV} = \frac{R_1 + 2 R_2}{V} = n \frac{R}{V},$$

or $R_1 + 2 R_2 = n (R_1 + R_2)$, from which

$$\frac{R_2}{R_1} = \frac{n - 1}{2 - n}. \qquad (9)$$

If we apply this to the case of a body obeying the normal law of skin-friction we have $n = 1{\cdot}5$, or $\frac{R_2}{R_1} = \frac{{\cdot}5}{{\cdot}5} = 1$, that is to say, the energy expended dynamically is equal to that expended in viscosity.

When the conditions are such that turbulence supervenes the expenditure of energy dynamically in the fluid disproportionately increases and consequently R_1 becomes greater than R_2, and in accordance with (9) the value of n rises, until for very high velocities it approximates more and more closely to 2, when the law becomes more nearly R varies as V^2.

The foregoing applies not only to the resistance of a plane moving tangentially through a fluid but to all cases of submerged fluid resistance; but at present the changes of the value of the index n have been but imperfectly investigated.

§ 41. Resistance-Velocity Curve.—Let us suppose that a curve *a, a, a, a* (Fig. 28) represents by its ordinates the resistance of a body of some particular geometrical form for different values of V (abscissae), which we may suppose have been determined experimentally; then if *b, b, b, b* be the curve for some other body of the same geometrical form but of *different linear proportions*, we shall, by the law of corresponding speeds, have for every given value of R, $V \propto \frac{1}{l}$, that is to say, the proportion $a\,c/b\,c$ is everywhere constant and the two curves are similar in relation to the axis of y. Also we have the relation $a\,c/b\,c$ in the inverse ratio of the respective linear dimensions, so that a single curve may be employed to represent the velocity-resistance relation of any given geometrical form, the velocity being read to a scale varying according to the linear dimension of the body, so that the diameter or some other definite linear dimension of the body is some definite and constant multiple or submultiple of the scale unit employed. Thus, if the curve be plotted for a one foot diameter circular plane and one foot per second velocity is represented by one inch, then for a two foot circular plane a one foot per second velocity will be represented by two inches, that is to say, a given pressure will be developed at one-half the velocity. This result is independent of the value of the index connecting R and V, or generally of the indices relating R, V_1 A_1 and V, of Equation (5); it would appear to be fundamental.

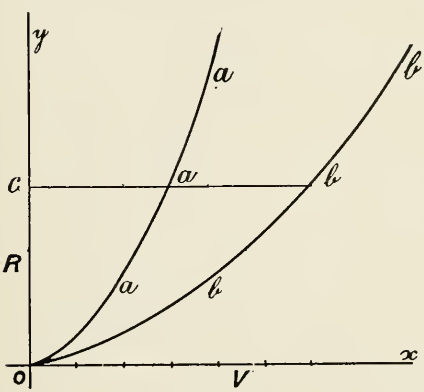

Fig. 28.

§ 42. Resistance - Linear Curve.

§ 42. Resistance - Linear Curve. — We may express the relationship of linear dimension and resistance directly in the form of a curve in which R is given by the ordinates as before, and l is represented by the abscissae, the curve being drawn for any given value of V. Now we have in Equation (5) $R = c\, v^q\, A^{\frac{r}{2}}\, V$, which we may write in the form $R = c\, v^q\, l^r\, V^r$, where l is a linear dimension on which A depends; in this form the expression is symmetrical in respect of l and V, and we have Equation (7) $V = c\,\frac{v}{l}$ also symmetrical with regard to these quantities, so that the form of the $R\ l$ curve will be identical with the $R\ V$ curve.

Thus let a, a, a, (Fig. 28) represent the $R\ V$ curve for a body of a certain geometrical form which we will entitle $(F)\ l$ where l is a linear dimension, then, for any value of l assigned to the body, there is for a given value of R a corresponding value of V such that Vl is constant; this is merely an expression of Equation (7) $Vl = c\,\nu$. But this holds good equally when the curve is read as an Rl curve, the value of the constant product of Vl being unaffected; consequently in reading the curve either as $R\,V$ or $R\,l$ the units are interchangeable. It may be noted that l may be the length, *i.e.*, the axial dimension of the body, or the transverse or some other dimension, without affecting the result, provided it is in all cases the same, and thus truly represents the linear size of the body.

§ 43. **Other Relations.**—In considering the relations of the curve of resistance we have hitherto taken the kinematic viscosity ν as constant; we will now study the consequences of taking this as a variable. So far the treatment has covered the case of variations of the velocity and linear dimensions of bodies in a fluid of constant physical properties; in supposing the viscosity to vary we are introducing the condition of a change of fluid, or at any rate such a change in the physical state of the fluid as is equivalent thereto.

Now, (7) $V = c\,\frac{\nu}{l}$ is the equation to *similar systems*, so that the similar system when ν varies is found when V or l or their product Vl varies in like ratio, that is the *scale value of the axis of x varies with the kinematic viscosity.* But by (5) $R = c\,\nu^q\,(lV)^r$, or for similar systems where $lV \propto \nu$, we have $R \propto \nu^q\,\nu^r$, where $q + r = 2$, that is, $R \propto \nu^2$, or the *scale value of the axis of y varies as the square of the kinematic viscosity.*

The conclusion may therefore be stated that:—*The resistance of a body of any definite geometrical form, in a stated aspect, may be represented as a function of its linear dimension (that is its size) and its velocity, by means of a single curve which may be termed its* characteristic curve of resistance, *the form of which is constant whether the abscissae represent linear dimension or velocity, and whatever the value of the kinematic viscosity may be.* And further, *in the interpretation of the curve the linear and velocity quantities alternatively represented by abscissae are interchangeable,* and *the scale value of the axis of x is proportional to the magnitude of the kinematic viscosity, and the scale value of the axis of y to the square of the kinematic viscosity.*

§ 44. **Form of Characteristic Curve.**—The form of the *characteristic curve of resistance* for different forms of body can, in all probability, only be determined experimentally. There are three ways by which the curve could be plotted, (*a*) by experimentally determining R for different values of V (or *vice versâ*), for any given body; (*b*) by the determination of the resistances of a number of bodies of geometrically similar form but of different scale dimensions, at any standard velocity; and, (*c*) by employing fluids of different viscosity and plotting indirectly, using a standard body at a standard velocity. The same curve should result in every case.

Of the three methods the last (*c*) may be dismissed as impracticable; the two former, (*a*) and (*b*), are, however, well suited to experimental conditions, and would furnish a complete check on the foregoing investigation. At present the experimental data are fragmentary and the evidence inconclusive.

The general properties of the curve, common to all forms of body, may be gathered from the circumstances of the problem. For very small values of V we know that quantities varying as $V^{1\cdot5}$ and V^2 become negligible, and the curve will be of the form $R \propto V$ and leave the origin as an inclined straight line. When the velocity is very great resistances that vary as the lower powers of the velocity will be negligible in comparison to those that vary as V^2, and consequently the curve will approach asymptotically to the form $R \propto V^2$. It is questionable whether the $R \propto V$ stage can exist when the viscous reaction of the *fluid is due wholly to its own inertia;* in the demonstration of the "normal law of skin-friction" it was shown that this condition results in the "1·5 power" law, and it would *appear probable* that this law is capable of more general demonstration, in which case the form of the curve (in an infinite region) will be limited to a *minimum index* of 1·5 and the straight line stage will disappear.

It is to be expected in any case that in actual experiment the ordinary viscous law will be found to apply for very low velocities on account of the fact that the size of the vessel containing fluid in which the experiment is performed cannot be made infinite, and for very low velocities the viscous stress or part thereof will be carried across to the walls of the vessel. Under these circumstances the condition that the reaction of the viscous substance shall be borne by its own inertia will not apply ; it is consequently of importance that experiments should be conducted in as large a tank as can be conveniently employed.

§ 45. Consequences of interchangeability of V and l.—It is evident that the general results relating to the form of the curve which have been deduced from the obvious relations of resistance and velocity apply to the less obvious relationship of resistance and linear dimension, owing to the interchangeability of V and l previously demonstrated (§ 42) ; we therefore see that—

For small similar bodies, *obeying the viscous law*, the resistance varies with the linear dimension, that is as the *square root of their area.*

For bodies of larger size, the resistance may be found to vary as the 1·5 power of the linear dimension, that is as the ·75 *power of the area.*

For bodies of very large size, the resistance will approach to vary as the linear dimension squared, that is directly as the area.

For planes moving tangentially it would appear possible that the latter condition is never attained but that some lower power may prove to be the limiting condition.

§ 46. Comparison of Theory with Experiment.—The foregoing theory receives substantial support from the experimental work of Froude, Dines, Allen and others.

In a series of experiments to determine the skin-friction of surfaces moving tangentially in sea water, Froude found that an increase in area is not accompanied by a proportionate increase in resistance; he also found that the index connecting resistance and velocity is in general less than 2, the mean result of several experiments giving 1·92. Colonel Beaufoy, also experimenting in sea water, gives the value 1·7 to 1·8.

Dines, experimenting in the open air, obtained results that have some interest from the present standpoint. In spite of some conflicting evidence, it would, in the main, appear that, under the conditions of experiment, the V^2 law is a very close approximation to the truth. In this Dines agrees with the previous experiments of Newton, Hutton, and others, and with the contemporary work of Langley.

It is to be inferred that in cases of direct resistance the Stokes ($R \propto V$) and Allen ($R \propto V^{1 \cdot 5}$) stages are confined to bodies of very small size and very low velocity. The bodies employed by Dines varied from some few square inches to some few square feet area.

Allen's work in connection with the present subject is of the greatest moment. The present application of dimensional theory is largely due to him as also its experimental verification. His investigations principally relate to spherical bodies of very small dimensions, and demonstrate *positively* that which has been already inferred *negatively*, *i.e.*, the small size and low velocities belonging to the Stokes and Allen stages of the characteristic curve.

§ 47. **Froude's Experiments.**—Owing to the condition of *constant geometrical form* not having been complied with in these experiments, some doubt exists as to the exactitude of the theory in its application. The planes employed differed in length alone, and it is evident that the skin-friction on a long narrow plane moving endwise will be less proportionately than one of more nearly square proportions, and consequently the effect of departing from the conditions will be to show a fictitiously low co-efficient for the longer of the planes employed.

If the width of the planes in proportion to their fore and aft length were sufficiently great, this effect would be negligible, as under these circumstances the sectional area of the fluid affected would vary substantially with the width itself; we will *provisionally assume* this to have been the case, and treat the fore and aft length as the l of the dimensional equation, at the same time bearing in mind the direction in which error is to be expected.

The first three series of experiments are as follows, the figure quoted being in each case the mean resistance per square foot taken over the whole area at a velocity of 10 feet per second :—

Nature of Surface.	Length of Surface (Fore and Aft Dimension).		
	2 feet.	8 feet.	20 feet.
Varnish	·41	·325	·278
Paraffin	·38	·314	·271
Tinfoil	·30	·278	·262

The values of the index calculated from the above observations, on the basis of the dimensional equation, are given in columns 1 and 2 of the Table that follows.

The observed index, that is to say the index calculated from observations made at *different velocities* is (taking the mean of all observations) given in column 3.

	Observation at 2 feet in relation to Observation		
	At 8 feet.	At 20 feet.	
	(1)	(2)	(3)
Varnish	1·825	1·832	1·88
Paraffin	1·86	1·854	1·94
Tinfoil	1·94	1·940	1·97

We here find the theory receives confirmation, inasmuch as, firstly, the order in which the indices arrange themselves is the same whether the computation is made, as by Froude, on the basis of experiments at different velocities, or as done here on the basis of change of linear dimension; and secondly, the quantitative discrepancy between columns 1, 2 and 3 is in the direction anticipated from the nature of the *provisional assumption.*

§ 48. Froude's Experiments (continued)—Roughened Surfaces.— When we examine the cases of *roughened surface* which form part of the series of experiments quoted, we find results that are not capable of such ready interpretation. In the case, for instance, of a surface coated with *coarse sand*, the index determined by Froude from experiments at different velocities was found not to differ sensibly from the maximum possible; that is, the index value is given as = 2. The constant velocity data in this case are:—

Surface.	2 feet.	8 feet.	20 feet.
Coarse sand ...	1·10	·714	·588

Calculating as before, we obtain the index values 1·69 and 1·728 respectively, which have no apparent resemblance to Froude's value. It is not possible to attribute this failure to the dissimilarity of geometrical proportion, for the previous calculations give an indication of the maximum value of the error introduced on this account; it is evident, therefore, that the cause must be sought elsewhere.

In the first three examples the nature of the surface is physically speaking *smooth*, that is to say, the roughness, such as it is, may be considered as *molecular*. Now we know in such a case that the drag produced on the fluid arises from the viscous connection between the film of the fluid actually contiguous to the surface and the strata more remote, and this connection—viscosity—is one fully taken account of in the equation; and even if the *molecular roughness* of one substance differs from that of another, the application of the theory will not be affected, When, however, we have to deal with a physical roughness, the conditions are altered, and in order that the theory should apply, the *scale of the roughness, i.e.*, the coarseness of the sand, must be increased as the length of the plane is increased; that is to say, the contour of the protuberances that constitute the roughness of surface *becomes part of the geometrical form of the body.* Thus, in the example quoted, the roughness, and so the resistance, is less on the 8 feet and 20 feet planes than it should be, and so the results are not comparable.

In all probability the difference between the values of the resistance for varnish, paraffin, and tinfoil is due to some difference in the physical roughness of these bodies, and so we shall expect to find the best agreement with theory in the case of tinfoil (which shows the smallest co-efficient); this is actually the case.

§ 49. Dines' Experiments.—The most suggestive experiments of Dines are those in which wind planes of different area are balanced about a vertical axis and the relative pressure so determined. Mr. Dines found that the pressure on normal planes does not increase in proportion to their area, but is proportionately greater on small than on large planes. The actual results obtained by observations on planes 6 ft. by 7 ft., 3 ft. by 3 ft., and 1 ft. 6 in. by 1 ft. 6 in., were that the pressure *per square foot* on a plane 6 ft. by 7 ft. is only 78 per cent. of that on one 3 ft. square, and that on the plane 3 ft. square is 89 per cent. of that on the 1 ft. 6 in. square plane. The actual velocity of the wind in which these experiments were made is not stated.

On the other hand, Mr. Dines specifically states that he finds the wind pressure on the normal plane and on bodies generally *varies strictly as the square of the velocity*, a result which it is difficult, in view of dimensional theory, to harmonise with the above experiments.

It is probable that the departure from the V^2 law is less than is indicated by the balanced plane experiment, owing to the smaller plane being unduly affected in each case by its proximity to the larger one. It is conceivable that the smaller plane *being situated in the counterwake of the larger*, will in effect be surrounded by air moving with above the normal wind velocity, and so show a fictitiously high pressure value. Mr. Dines' elegant method of determining the V^2 law, by balancing against centrifugal force, would appear to be quite above suspicion, although it may not be sufficiently sensitive to demonstrate the departure from the law, which for the normal plane is certainly very small indeed. In any case the results, without some such explanation as given, are not altogether consistent, and a repetition of these experiments ought to be made.

§ 50. Allen's Experiments.[1]—Mr. H. S. Allen, experimenting with bubbles and small solid spheres in liquids, found that for very small velocities the viscous law holds good, whereas for very great velocities the V^2 law prevails; he also shows that there is an intermediate well-defined range, over which the $V^{1\cdot5}$ law applies. His results are summarised as follows:—

"Three distinct stages have been recognised:

"(1) When the velocity is sufficiently small the motion agrees with that deduced theoretically by Stokes for non-sinuous motion, on the assumption that no slipping occurs at the boundary; in such motion the resistance is proportional to the velocity.

"(2) When the velocity is greater than a definite critical value, the terminal velocity of small bubbles and solid spheres is proportional to the radius, less a small constant; it may be expressed by the formula given.

"(3) For velocities considerably greater than those just considered, the law of resistance is that which Sir Isaac Newton deduced from his experiments, namely, that the resistance is proportional to the square of the velocity."

Of the above three stages, (2) corresponds approximately to the

[1] *Phil. Mag.*, September and November, 1900.

law, *resistance varies as the* 1·5*th power of the velocity,* for, where the force overcoming the resistance is supplied by the difference of specific gravity of the fluid and the sphere, we have—

$$R \propto l^3,$$

and $$R \propto V^r\, l^r\, \nu^{2-r},$$

$$\therefore \quad l^3 \propto V^r\, l^r\, \nu^{2-r},$$

$$\therefore \quad l^{3-r} \propto V^r\, \nu^{2-r},$$

that is, $$V \propto \frac{l^{\frac{3-r}{r}}}{\nu^{\frac{2-r}{r}}}.$$

But for a given fluid ν is constant, and we have—

$$V \propto l^{\frac{3-r}{r}},$$

or if $$V \propto l^n \qquad n = \frac{3-r}{r}, \quad \text{or} \quad r = \frac{3}{n+1}.$$

Now, if in stage (2) (Fig. 29) we ignore the small constant, we have $V \propto l$ or $n = 1$, and $r = \frac{3}{2}$, that is to say, the general expression in this case becomes:—$R = c\, \nu^{\frac{1}{2}}\, l^{1\frac{1}{2}}\, V^{1\frac{1}{2}}$, that is, during this stage the resistance follows the *normal law of skin-friction, or Allen's law.*

§ 51. Characteristic Curve, Spherical Body.—The form of the experimental curve, as plotted by Mr. Allen, is given in Fig. 29, in which ordinates = V, and abscissae = values of linear dimension, *i.e.*, radius of sphere. The first stage or Stokes portion of this curve is a parabola, $V \propto l^2$; this corresponds to an r value = unity; the second stage is approximately a straight line, the value of r here being as shown 1·5; the third (or Newtonian) stage of the curve, not shown on this plotting, has an r value equal 2, that is $n = {}·5$ or $V \propto l^{·5}$. This form of plotting is the outcome of the method of experiment, *i.e.*, measuring the limiting velocity acquired under the influence of gravity; if we re-plot as a resistance-velocity diagram (Fig. 30), the size of the body being supposed constant, we are able to obtain a general idea of the "characteristic curve of resistance" for a spherical body. In Figs. 29 and 30 the curve does not extend to the V^2 stage; in Fig. 31 we have the three stages represented *diagrammatically*, firstly, by a straight line departure from the origin where $R \propto V$—this is the *Stokes stage;* next we have a section of the curve following the $R \propto V^{1 \cdot 5}$ law, the *Allen stage*, and lastly the curve will approximate to a parabola where $R \propto V^2$. The latter stage is that investigated both theoretically and experimentally by Newton ("Principia," Book II., Section VII.), determinations being made both in water and air; also in the year

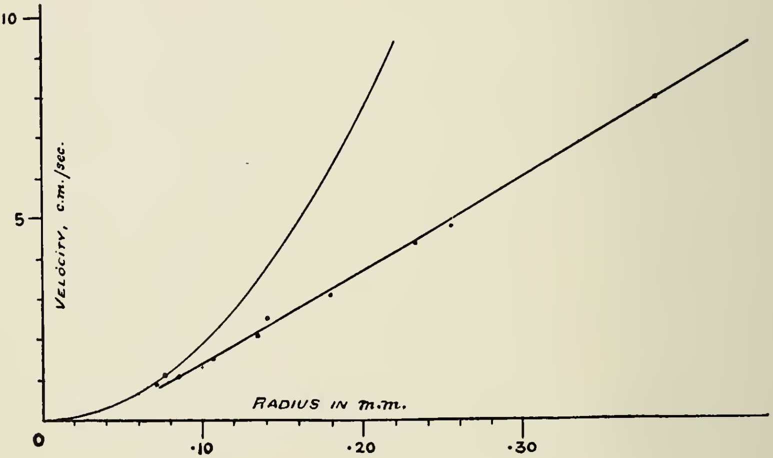

FIG. 29.

1719 by Dr. Desaguliers, who employed spherical bladders let fall from the cupola of St. Paul's. Newton's theoretical investigations were based on the hypothetical medium of discrete particles, but the experimental verification was sufficiently close, *qualitatively*, to establish the velocity squared law as substantially correct, so far at least as the sizes of sphere and velocities employed in his own and Desaguliers' experiments are concerned.

§ 52. **Physical Meaning of Change of Index.**—The nature of the alteration in the system of disturbance that accompanies each change of "law" is a matter of considerable interest. The

Stokes law is based on a system of motion of the fluid that has been mathematically investigated and the lines of flow plotted from an equation.[1] It may be remarked that this system of motion can never exist in its entirety, for it involves an infinite

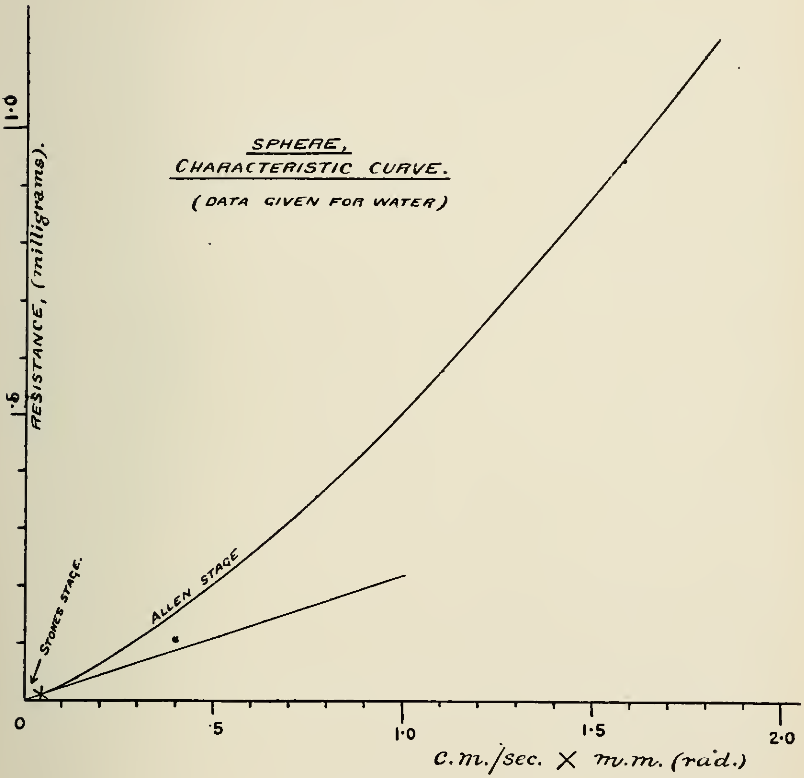

Fig. 30.

quantity of momentum and an infinite quantity of energy[2]; in other words, the steady state involves a force applied for an infinite time through an infinite distance; it also constitutes a violation of the *principle of no momentum* of § 5.

If we suppose that the stress, due to the propulsion or to the

[1] Stokes' Scientific Papers (t. iii.).

[2] "Hydrodynamics," H. Lamb, 1906 ed., p. 553.

resistance of the body, be transmitted *by viscosity* to the walls of the vessel, as when the body is moving quite slowly, or when the thickness of intervening fluid is small; then the resistance will

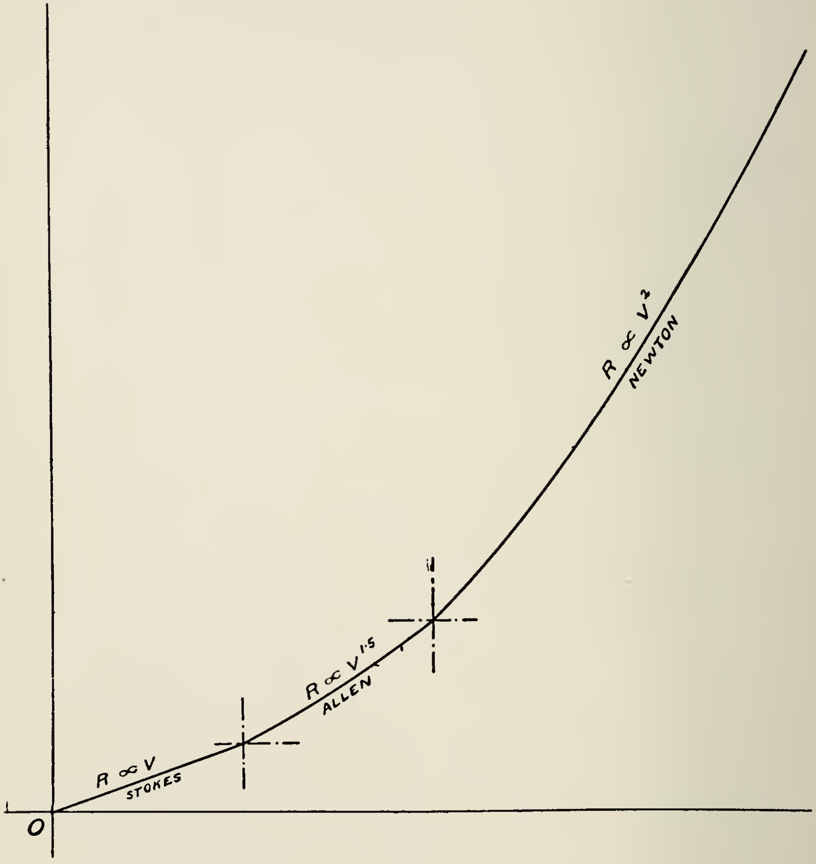

Fig. 31.

evidently follow the ordinary viscous law. When, however, the viscous drag is resisted by the inertia of the fluid, that is to say, there is no continuity of viscous stress from the body to the walls of the vessel, then it would appear probable that the law of skin-friction applies. If this view is correct, the extent of the stage where R varies as V will depend, as already stated, upon the size of the containing vessel.

§ 53. Change in Index Value (continued).—We have so far confined ourselves to the discussion of the first change of index, that which takes place when the curve passes from the Stokes to the Allen stage.

The second change of index value evidently takes place when the motion of the fluid becomes turbulent, for it is then that the conditions leading to the *normal law of skin friction* are violated, and the energy relation becomes disturbed. In all probability also the V^2 law, in cases involving other than pure skin-friction, is closely associated with the phenomenon of *discontinuity.* A system of flow of the discontinuous type is almost certainly accompanied by resistance following the V^2 law.

The conclusions of this and § 52 at present lack experimental demonstration. There would appear to be some evidence to show that the Stokes stage may exist independently of the size of the vessel; this at least is a conclusion reached by Allen. If this should prove to be the case the explanation here given will need modification.

§ 54. The Transition Stages of the Characteristic Curve.—The junction or transition portions of the curve connecting the various stages are not angular as shown diagrammatically in Fig. 31, but pass gradually from the one to the other. The transition stages, however, are not such as to mask the distinct individuality of each portion of the curve, but merely enough to render uncertain the precise point at which the change of "law" takes place.

It appears that there is a small departure from the exact expression given, both in the second stage (as found by Mr. Allen) and in the third stage. In the latter case we may suppose that when the velocity becomes very great the geometrical form of the lines of flow becomes sensibly constant, and such resistance as is due to viscosity will then vary as the velocity, the net resistance curve being thus the sum of the ordinates of a parabola

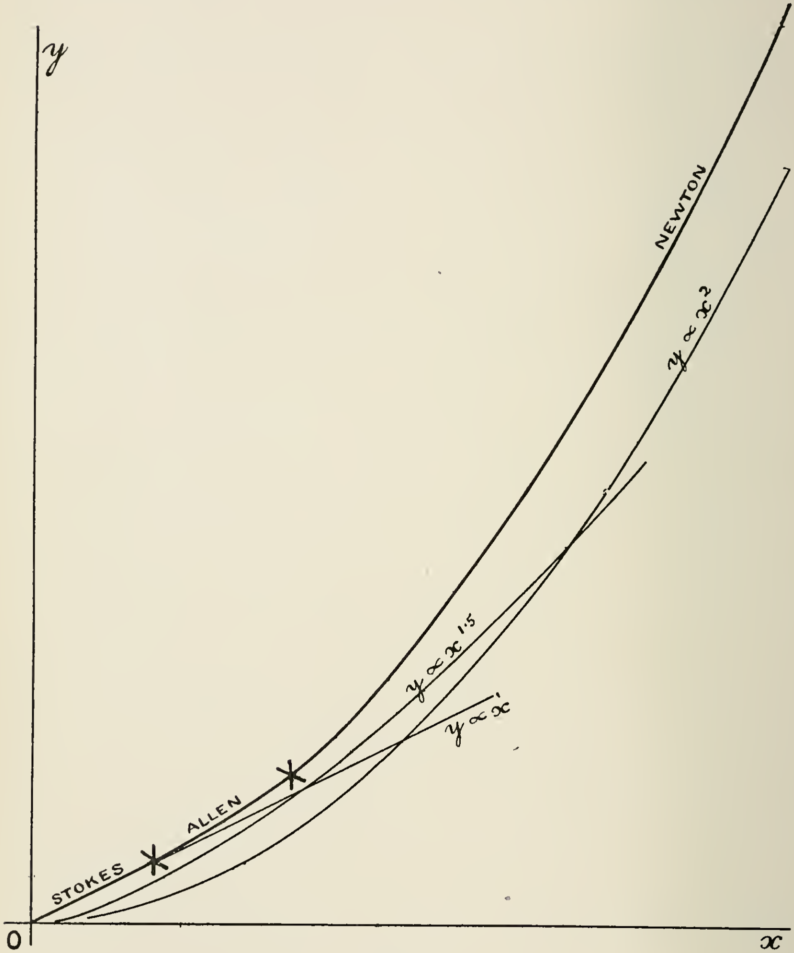

FIG. 32.

and an inclined straight line. It can be shown geometrically that this results in the curve of resistance approximating to a *parallel* to the true parabola as shown in Fig. 32. In the second stage we have the experimental result of Mr. Allen as a guide. We know that if the second stage law were to hold good down to zero velocity, we should have a certain small residuary resistance (see Fig. 29). This means on the velocity-resistance diagram that the *origin* for the 1·5 index curve will be situated a short distance up the axis of y; we may conveniently construe this as an approximate parallel to the curve struck from the true origin, when Fig. 32 will represent the manner in which the resistance curve may be supposed to be built up.

§ 55. Some Difficulties of Theory.—In all cases of skin-friction where the index exceeds 1·5 the motion is accompanied by turbulence, and if the value of the index rises to 2, as it would appear to do approximately in the case of the roughened surface, then the dimensional equation (as pointed out by Allen) shows that the resistance is independent of viscosity, and the whole of the energy is expended dynamically in producing fluid motion. Under these circumstances we must regard viscosity as merely acting as a *gearing* by which rotational motion is imparted to the fluid, although it is difficult to understand how such a gearing can be continually imparting rotation to new masses of fluid without a certain amount of *slip*; and such slip would betoken an expenditure of energy in viscous motion and necessitate the value of the index being less than 2.

Beyond this it is difficult to conceive of the resistance being independent of the *value* of viscosity without being independent of the *existence* of viscosity, which appears to be absurd. Consequently it is probable that, *so long as the effect of elasticity of the fluid is not felt*, the value of the index, connecting resistance and velocity or resistance and linear dimension, can never reach its limiting value, 2, but must always fall short of it by some small quantity.

It is known, from experiments on resistance in the flight of projectiles, that for velocities approaching the velocity of sound, the index may rise considerably above the limiting value given in the foregoing theory; and therefore we may expect to find in general that the experimental determinations, except for velocities quite small in comparison with that of wave motion, will be in excess of those indicated by the theory. It is evident that in *elasticity* we have a factor foreign to the dimensional theory as given, and the existence of such a factor invalidates the hypothesis upon which the theory is founded.

An apparent discrepancy occurs in the case of some experiments made by Newton (Book II., Section VI.), who found, from the motion of a pendulum whose spherical "bob" was immersed in water, that the *resistance was augmented in more than the duplicate ratio of the velocity.*

Newton supposed this to be an error due to the *narrowness of the trough* employed, but this in the light of dimensional theory is insufficient.

The probable explanation is that for a large arc the discontinuous type of motion has time to establish itself on each swing, whereas for small arcs of motion the flow has not had time to fully develop discontinuity; for very small arcs the flow will approximate to the Eulerian form (compare Chap. III.). Consequently the resistance for small amplitude is far less than is the case for continuous motion, and thus factors are introduced outside the dimensional hypothesis, which presumes a steady state.

§ 56. General Conclusions.—The importance of the results attained in the present chapter, in relation to aerial flight, is to some extent an unknown quantity.

It is evident that under ordinary conditions the *law of viscosity* does not apply, and it would appear further that the tangential resistance does not follow the *normal law* of *skin-friction,* but that the conditions commonly involve *turbulence*, and R varies as some higher power of V between $V^{1 \cdot 5}$ and V^2. It is highly probable that the conditions may be different in the case of the smaller flying insects, such as flies, mosquitoes, etc., and it may be the relatively greater importance of viscosity in such cases that is primarily responsible for the peculiarities of insect flight.

For large birds and flying machines the "*R varies as* V^2" law is probably accurate enough for ordinary computations of resistance, whether *frictional* or *direct.* The V^2 law is generally assumed in the present work as sufficiently near the truth; the assumption of one law for both classes of resistance results in a simplification of method which fully justifies its employment, even at the expense of some small degree of accuracy.

The meaning of the statement (§ 36) that viscosity gives a *scale* to the fluid, may be illustrated by supposing a "bluebottle" to find itself transformed into a common fly (supposing the two to be strictly proportional in their parts): it would find that the *apparent* viscosity of the air had increased; in other words, the air would appear to be more "sticky" than usual. The same fact is familiar in other cases—for example, the difference in character of a large and a small flame, etc. Other physical properties are capable of giving a *scale* to a fluid: thus *elasticity* as demonstrated by the length traversed by a wave in unit time; *surface tension* as demonstrated by the velocity of *slowest surface wave.*

One of the least satisfactory results of dimensional theory, so far as revealed by a comparison with conclusions that would be naturally formed from experience, is the inverse relation that exists for homomorphous motion between V and l. It would appear that for bodies of similar form any *state* of *motion*—say the state when discontinuity sets in—is reached when their respective velocities are in the inverse ratio of the linear dimension. Thus, if a salmon and a herring were geometrically proportional, the herring would be capable of a higher velocity, without ceasing to be of streamline form (by definition), than the salmon in the inverse proportion of their respective lengths. Now this seems very unsatisfactory, for a whale would be scarcely capable of locomotion without carrying a deadwater region in its wake—a most improbable conclusion. Certain explanations are possible, but the author has been unable up to the present to find any conclusive solution to the difficulty.

CHAPTER III.

THE HYDRODYNAMICS OF ANALYTICAL THEORY.

§ 57. **Introductory.**—The analytical treatment of hydrodynamic problems commonly involves an extensive application of the higher mathematics, the classic methods being those of Euler, Lagrange, Stokes, and others.

The importance and bearing of the *mathematical demonstrations*, in connection with the subject of the present work, is comparatively limited, but many of the results are of great consequence; the present exposition has therefore been restricted to a brief indication of the mathematical method, and a digest of those results which, from the present standpoint, are of the greatest interest. Where it has been found possible, a simple physical demonstration is given; in many cases the results of established investigation are taken for granted.

The present discussion opens with a *recapitulation* of the physical properties of fluids, which may be taken as a concise re-statement of essential definitions, sufficient to render back reference unnecessary. The chapter concludes with a critical argument on the practical deficiencies of the *Eulerian* and *Lagrangian* method, and on the theory of Discontinuous Flow.

The hypothesis of the initial discussion is strictly that of an *inviscid* fluid, and in general the condition of *incompressibility* is assumed. Up to a certain point the mathematical treatment, as usually applied, takes cognisance of compressibility, but generally speaking, the tangible results, so far as they concern our present subject, relate to the simpler conditions.

§ 58. **Properties of a Fluid.**—All fluids are characterised by certain definite physical properties. The property that may be said to constitute fluidity, and which distinguishes fluid from solid bodies, is *inability to sustain stress in shear.* A fluid in which this property is perfect is said to be *inviscid*, and in such a fluid a shearing strain, *i.e.*, *distortion*, may take place without being accompanied by any corresponding stress: such a fluid must be regarded as hypothetical. All actual fluids possess viscosity; in a viscous fluid a stress in shear may exist, but is accompanied by a continually increasing strain; stress in shear in a viscous fluid bears, in fact, the same relation to the *rate of change of strain* that stress in a perfectly elastic solid bears to the strain itself.

The remaining physical properties of a fluid are identical with those of a solid body, and comprise *density* and *elasticity* (volumetric). These two quantities are related to a third quantity—pressure—in so much that the density is a function of the pressure, the nature of which function is defined by the *law of elasticity;* thus in a perfect gas under isothermal conditions we have $\frac{P}{\rho}$ = constant where ρ is density, and P pressure.

If we take the two extreme cases in the relation of ρ and P, so that, firstly, the elasticity be supposed zero, we shall have any finite pressure, however small, produce an infinite density, and the fluid becomes identical with the *medium* of Newton. If, secondly, we suppose the elasticity to be infinite, so that a change in P, however great, produces no change in the density, we have the case of an incompressible fluid. The latter assumption is that of our present hypothesis.

§ 59. Basis of Mathematical Investigation.—The *Equations of Motion* may be said to constitute the starting point of all analytical investigation; these are:—

(1) The *Equation of Continuity*, expressing the relation between the density of the fluid and the linear rate of change of flow in each of the co-ordinate directions of space; or, under the restriction that density is constant, the relation between the

(linear) rate of change in the three co-ordinate directions amongst themselves.

The equation of continuity is based upon the fact that the inflow and outflow of any small element of space must balance, or must balance against the change of density if the fluid is compressible. The form of the expression for an incompressible fluid is—

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

where u, v, and w represent the velocities in the directions of the three co-ordinate axes x, y, and z.

(2) The *Dynamical Equations* expressing the relation in the direction of each of the three co-ordinate axes x, y, z, for every small element of the fluid, between the rate of change in its momentum, the difference of pressure on its opposite faces, and the component of the extraneous force, if any.

The Extraneous Forces are usually represented in the three co-ordinate directions by the symbols X, Y, Z, and denote forces acting from without on the fluid particles, such, for example, as the force of gravity. In the present branch of the subject these forces do not require to be considered.

Employing, as is customary, the symbol DF/Dt to denote a differentiation following the motion of the fluid, it can be shown that

$$\frac{DF}{Dt} = \frac{dF}{dt} + u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz}. \qquad (1)$$

Now the rate of change of the x momentum of any small element $\delta x\ \delta y\ \delta z$ is $\rho\delta x\ \delta y\ \delta z\ \frac{Du}{Dt}$, and this must be equal to the difference of the pressure force on its two faces, which is evidently $-\frac{dp}{dx}\delta x\ \delta y\ \delta z$ (where p is pressure). The *minus* sign is due to the fact that the momentum increase takes place in the direction of the pressure decrease. So that :—

$$\rho\delta x\ \delta y\ \delta z\ \frac{Du}{Dt} = -\frac{dp}{dx}\delta x\ \delta y\ \delta z,$$

or

$$\frac{Du}{Dt} = -\frac{dp}{\rho dx}.$$

Substituting from (1) for Du/Dt, Dv/Dt, and Dw/Dt we have:—

$$\left.\begin{aligned} \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} &= -\frac{dp}{\rho dx}, \\ \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} &= -\frac{dp}{\rho dy}, \\ \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} &= -\frac{dp}{\rho dz}. \end{aligned}\right\}$$

In the steady state du/dt is zero, and the equations become:—

$$u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} = -\frac{dp}{\rho dx}, \text{ etc.}$$

When there is no motion or motion of translation only in the fluid, the last three terms of the left-hand side of the equation are zero, and the equations become:—

$$\frac{du}{dt} = -\frac{dp}{\rho dx}, \text{ etc.}$$

The further development and employment of these equations is outside the scope of the present work, but the physical significance can be gathered by comparison with §§ 60 and 88.

The mathematical superstructure founded on the above consists in the main of finding solutions to the equations of motion in a number of well-defined cases, and in the general development of the theory in its application to the motions of bodies of stated geometrical form under known boundary conditions.[1]

§ 60. Velocity Potential (ϕ Function).—If a force be applied to a body initially at rest in a fluid, a circulation of the fluid is set up, the flow taking place along paths of curvilinear form by which the displaced fluid is conveyed from one side of the body to the other. We may regard the initial direction of flow, produced in this manner, as denoting a "field of force," the

[1] For the full mathematical treatment reference should be made to "Hydrodynamics," H. Lamb, Camb. University Press.

direction of the *lines of force* being everywhere that of the initial acceleration of the particles.

When such a system is initiated in a fluid from rest, at the instant the force is applied the surfaces of equal pressure are everywhere normal to the lines of force. This is not necessarily the case when the fluid is in motion, for we have then superposed pressure differences due to the change in velocity and direction of the particles which modify the pressure distribution.

Let us suppose that the applied force is impulsive, *i.e.*, let it be considered to be an infinite force applied for an infinitely short time; then the form of flow generated will be that due to the initial application of the force, that is to say the field of flow will coincide with the field of force.

Now it does not obviously follow that this form of flow will be stable or permanent. In actual fluids, such as water or air, we know in fact that it is not so. It would, however, appear that in the ideal fluid of hypothesis any form of motion generated by an impulse in this manner will persist without change of form, and therefore the field of force and system of pressure by which the flow is generated may be taken as defining the form of flow for the steady state.

Under these circumstances it is evident that the motion will be the same whether generated by an impulse or by a finite force, since the continued application of the force to the body in motion will accelerate the field everywhere in the line of flow.

If we now examine the initial pressure system, then the velocity produced on the fluid from rest along any line of force after a brief interval of time will be, for any small difference of pressure, inversely as the mass per unit section, that is, inversely as the distance separating the points at which the said pressure difference exists. Or if δp is the pressure increment, and δl the distance along the line of force, the velocity after a certain brief interval of time will be everywhere proportional to $\frac{\delta p}{\delta l}$, or, when the increments are taken as evanescent velocity varies as $\frac{dp}{dl}$, or, resolving into its three co-ordinate components, we have—

$$u, v, w, = c\frac{dp}{dx}, \ c\frac{dp}{dy}, \ c\frac{dp}{dz},$$

where c is a constant.

In the above expression the density of the fluid and the magnitude of the applied force are involved in the constant c. It is, however, evident that we may regard the form of flow as a matter of pure kinematics, since the existence of the flow is not dependent upon the pressure system by which it is generated. Consequently we may substitute for p a function ϕ, which has no dynamic import, and which is termed *velocity potential*, and we may write the expression—

$$u, v, w, = \frac{d\phi}{dx}, \ \frac{d\phi}{dy}, \ \frac{d\phi}{dz},$$

the terms on the right-hand side of this equation being sometimes written with a *minus* sign.

In the foregoing illustration ϕ is a *single-valued* function, inasmuch as it can have a definite value assigned for every point in the field of flow.

§ 61. Flux (ψ Function), ϕ and ψ interchangeable.—In cases of fluid motion in which a velocity potential exists the lines of flow are, as pointed out, everywhere normal to the equipotentials, that is to say the surfaces of ϕ = constant. It can be shown analytically that if the curves of flow be plotted for equal increments of *flux* (that is, so that the amount of fluid that flows per unit time past any point and between two adjacent lines is constant), and the curves ϕ = constant be plotted over the same field, the two series of lines will divide the field into a number of similar elements whose ultimate form in the case of motion in two dimensions, when the units employed are sufficiently small, becomes square within any desired degree of approximation. Thus (Fig. 33) let c e be two lines of flow, and f f be two lines

ϕ = constant; then the cell cut off, *a*, *b*, *c*, *d*, will be approximately square, and if we choose to subdivide for intermediate values of flux and velocity potential as indicated, the cellules so formed will approximate still more closely, and the whole field may be regarded as ultimately built up of a number of such square elements.

If in two-dimensional motion the successive increments of flux be represented by increments of a quantity ψ, it can be shown that the ϕ lines and the ψ lines may be interchanged, the lines of equal flux becoming equipotentials, and *vice versâ*. The applied impulse will of course require to be different for the two systems.

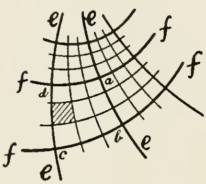

FIG. 33.

When the motion takes place in three dimensions, the ψ lines and ϕ surfaces still divide the fields into a number of rectangular elements, or cubes (Fig. 34), but the conjugate property no longer exists; the ψ, ϕ, functions are not interchangeable.

The foregoing principles may be illustrated by the simple case of a *source* and *sink*.

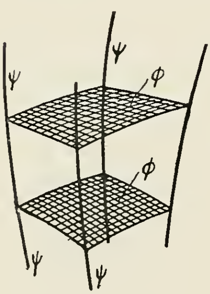

FIG. 34.

§ 62. Sources and Sinks.—A *source* is a hypothetical conception, and may be defined as a point at which fluid is being continuously generated, and conversely a *sink* is a point at which fluid is supposed to disappear. Nothing actually resembling a source or sink is known to experience, the utility of the conception resting in its application to theory. A *point source* gives rise to three-dimensional motion; a *line source* gives rise to two-dimensional motion. A line source may also be described as a *point source in two dimensions*.

The field of flow from a source or towards a sink in an infinite expanse of fluid can be laid down from considerations of symmetry. The conditions require that it should be constituted of radial straight line flow equally distributed circumferentially in space. The field, in the case of motion in two dimensions, being shown plotted in Fig. 35 for a series of equal increments of flux, each line of flow will represent some definite value of the function ψ, any one of the lines being arbitrarily chosen as datum.

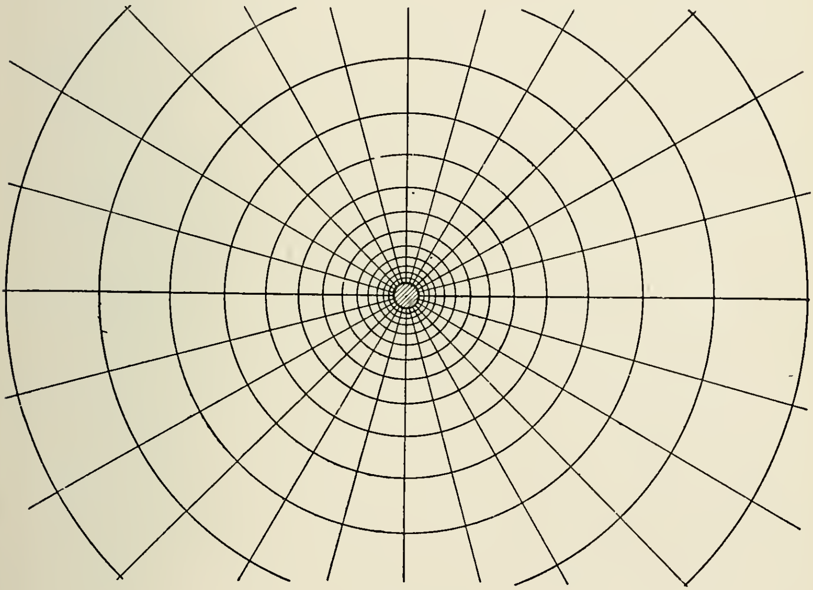

FIG. 35.

The equipotentials, ϕ = constant, will be a series of concentric circles whose radii form a geometrical progression.

If the functions ψ and ϕ be interchanged, the diagram represents a *cyclic motion round a filament, the radial lines becoming the equipotentials.*

In the case of the source or sink, the velocity of the fluid at the origin is infinite, the whole flux having to pass through a region having no magnitude. In order to keep the problem within the range of physical conception it is customary in this and similar cases to suppose the source or sink to be circumscribed by a small closed curve, which in the case we have under consideration will be a circle. When we interchange the functions ψ and ϕ the same considerations apply. In this case the space within the circular enclosure represents the section of a cylindrical filament, around which the cyclic motion of the fluid is taking place. The introduction of such an obstacle, *i.e.*, a circumscribed area in a two-dimensional space or an infinite cylinder in a three-dimensional space, involves what is termed the *connectivity* of the region. Where no obstacle exists the region is said to be *simply connected;* where one or more such obstacles exist the region is *multiply* connected. The question involves certain points of definition.

§ 63. Connectivity.—It is possible to connect any two points in a region containing fluid by an infinite number of paths traversing the fluid. Such paths as can by continuous variation be made to coincide without passing out of the region are said to be *mutually reconcilable.*

Any circuit that can be contracted to a point without passing out of the region is said to be *reducible.*

Two reconcilable paths combined form a reducible circuit.

A *simply connected region* is one in which all paths joining any two points are reconcilable, or such that all circuits drawn within the region are reducible.

A *doubly connected region* is one in which two irreconcilable paths, and no more, can be drawn between any two points lying within it, so that any third path shall be reconcilable with the one or the other, or shall be in part reconcilable with one or the other, and in part reducible to the circuit formed by the two combined. (The latter portion of this definition is necessary to provide for the case of a third path being drawn making one or more circuits of the "obstacle.")

In general, multiply-connected regions, in which n irreducible paths, and no more, can be drawn to connect any two points, are said to be *n-ply connected.*

A few examples may be given. The region internal or external to the surface of a chain link or an anchor ring is a doubly connected region ; a simple electric circuit, either internal or external to the conductor, is a doubly connected region; on breaking the circuit both regions become simply connected. The cavity of the labyrinth of the human ear is a triply connected region, as also is a lake containing two islands. The region surrounding a gridiron is n-ply connected where n is the number of the bars.

§ 64. Cyclic Motion.—The subject of connectivity derives its importance chiefly from its relation to the class of fluid motions known as *cyclic.* In a simply connected region, for all motions having a velocity potential, the latter, ϕ, is a *single-valued* function, having at every point in the system a definite assignable value, varying continuously from point to point throughout the system. When the region is doubly connected this manifestly *may* not be the case, for if there is a circulation around an irreducible circuit it is evident that if we follow the variation of ϕ round such circuit we shall on arriving at the starting point have two conflicting values. Thus, referring to Fig. 35 when the radial lines are taken to represent ϕ = constant, we are unable to assign a progressive series of values to the ϕ lines that will be consistent.

Under these conditions ϕ is termed a *cyclic function,* and its value depends upon the datum point chosen for its zero and the number of times the path of integration has been taken round a circuit.

A physical conception of velocity potential under these circumstances is somewhat difficult, but if we revert to the dynamical hypothesis and regard the velocity potential system as the pressure system by which the motion is generated, we encounter at once the same difficulty in another form. Before we are able to interfere either to start or to stop the fluid in cyclic motion, we must introduce some imaginary barrier in its path. In the special case in the figure this evidently requires to extend from the central core outward to infinity in order to intercept the whole of the flux. We are at liberty to select what position we like, circumferentially, for this barrier, and in choosing such position we fix the datum for the value of ϕ. If then we suppose a suitable impulse to be applied the p of the impulse pressure system will be a single-valued function throughout the field, and p defines ϕ for the subsequent motion. The barrier, however, cannot be maintained under the conditions of steady motion, and it is the *withdrawal* of the barrier that renders ϕ indeterminate. It is the complementary fact that the barrier temporarily renders the region simply connected, and on its withdrawal the cyclic conditions supervene.

The particular case of cyclic motion taken as an illustration is one of the most elementary simplicity. The degree of complexity of any cyclic system of flow depends primarily upon the boundary conditions. We shall have occasion to refer later to cyclic systems of greater complexity, but at present the complete solution of the equations of motion is only known in some few cases where the boundary conditions are simple.

Although in any case of cyclic flow, such as in the example given, the fluid is in circulation around a central island, and so as a whole possesses angular momentum and rotary motion in the ordinary acceptation of the words, such a form of flow (*i.e.*, one that can be generated by an impulse and possesses a velocity potential) is in reality *irrotational.* The theory of *rotation* in fluids is of considerable importance, in view of the fact that it can be proved that if the motion of an inviscid fluid is irrotational at any instant of time, it will remain irrotational for all time; that is to say, it is impossible to produce or destroy rotation in an ideal fluid.

§ 65. Fluid Rotation. Conservation of Rotation.—Let us suppose a hollow circular cylindrical vessel filled with fluid to be set in rotation about its axis: then if the fluid possessed viscosity it would, in course of time, acquire sensibly the same speed of rotation as the vessel, so that the whole system would revolve *en bloc*. With an ideal fluid, however, the rotation of the vessel might be continued indefinitely without imparting any motion to its contents.

If we suppose substituted for the circular cylinder one of square or other irregular section, it might be imagined that rotation would be imparted to the fluid by the irregularity of the boundary surfaces; such, however, is not the case. An inviscid fluid offers no resistance to *distortion*, and consequently the containing vessel, however irregular its form, is unable to acquire a "purchase" on the fluid contents, and the fluid is not set in rotation. Conversely, if we suppose the fluid to be in a state of rotation in a vessel or region, no matter what its form, such rotation will persist and the fluid will continue to rotate for an indefinite time.

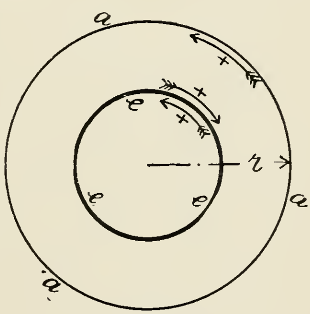

Fig. 36.

The foregoing reasoning, although touching the essence of the matter, can hardly be regarded as rigid proof.[1]

§ 66. Boundary Circulation the Measure of Rotation.—The study of rotation may be confined to two dimensions. Let $a\ a$ (Fig. 36) represent a circular cylindrical vessel of radius r within which the fluid possesses a motion of pure uniform rotation.

Now, such rotation is shared uniformly over the whole area; therefore, if we suppose the area divided into a number of equal small elements, and represent the rotation of each by a circulation

[1] The mathematical demonstration of this important fact will be found in "Hydrodynamics" (H. Lamb, Cambridge), or reference may be made to the original investigation (Lagrange, "Oeuvres," T. IV., p. 714).

round its boundary (Fig. 37), then the circulation round each element will be equal, and that along all the lines common to two adjacent elements is equal and opposite, and therefore of zero value, so that circulation along the boundary alone remains. It is proved then that:—

The sum of the circulations round the boundaries of the individual elements is equal to the circulation round the boundary of the region; that is to say, *the rotation of the fluid within the region is measured by the circulation round its boundary.* It is evident that this result is not confined to uniform rotation. Let us suppose that the fluid contain rotation unevenly distributed amongst its parts, so that it may be in part irrotational, and in parts the sense of rotation may be opposite to that in other parts, but so that the velocity (u v) is, throughout the region, a *continuous function* of x y; then if we suppose it be divided as before into a number of small elements so that each element shall be indefinitely small, then the rotation within each element is uniform, and by the preceding argument is measured by the circulation round its boundary; but since u v is a continuous function of x y, the flow along the boundary of each element is in the limit equal and opposite to that of the element adjacent to it, and the two cancel out, leaving only the circulation round the boundary. Hence for any region the sum of the rotation integrated over the surface is equal to the sum of the circulation integrated along its boundary.

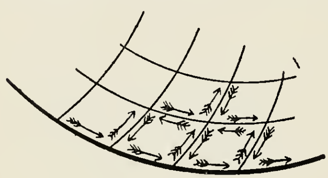

Fig. 37.

§ 67. Boundary Circulation Positive and Negative.—Referring again to Fig. 36, let us suppose a boundary surface to exist at e e e dividing the region into two parts, and let e e e coincide with one of the lines of flow so that it will not interfere with the motion of the fluid; the boundary e e will thus be circular, and concentric to the boundary a a.

Then if r be the radius of the whole enclosure *a a*, and $n\ r$ be the radius of the region *e e*, and χ the total rotation, the rotation within the region *e e* as measured by the circulation along its boundary will be $n^2\ \chi$; the remaining rotation in the region between the boundaries will therefore be $\chi - n^2\ \chi$, that is to say, the circulation along the external surface of the boundary *e e* is equal and *opposite in sign* to that along its internal surface.

Now if we regard the rotation of the fluid mass as a matter of rigid dynamics, the motion in the path *e e* is the same in sense whether it takes place in the matter external or internal, and in general rotation is an algebraic quantity, measured *plus* or *minus*, according to whether it takes place counterclockwise or clockwise (the latter being taken *minus* by convention). It is evident, however, that *circulation along a boundary* (also an algebraic quantity) cannot be so measured, but is *plus* or *minus* according as the fluid flows towards the right or the left hand of an observer stationed on the boundary facing the fluid. Thus, in the simple case illustrated, let us suppose the rotation to be positive (counterclockwise), then to an observer stationed on the "mainland" the circulation will pass from left to right, and is reckoned positive. If the observer now take his stand on the "reef" *e e*, and face the outer basin, the circulation will pass from right to left, and is therefore negative. If he now turn about and face the inner basin, the circulation is from left to right, and is positive. Another method of defining the *sense* of a circulation is to suppose an observer swimming in the fluid to keep the boundary always on his right hand, then the direction in which he is swimming is positive and the opposite direction negative. The positive direction is indicated by arrows in the figure.

Rotation in a fluid as above defined is a conception apart from any quantity known in rigid dynamics, and owes its importance to certain propositions relating to fluid motion. It is a quantity that in a perfect fluid can undergo no change. Conservation of rotation is an absolute law in an inviscid fluid.

§ 68. Rotation. Irregular Distribution. Irrotation,—Definition.—The propositions connecting boundary circulation and rotation include all cases of rotation, so that we know that however much the rotation differs in different parts of the fluid, the algebraic sum of the rotation taken over the whole of the region is equal to the integration of the circulation along the boundary (reckoned *plus* or *minus*, according to the law laid down).

Thus, *the total rotation in a region containing fluid is zero when the sum of the circulation taken over a complete circuit of the boundary is zero;* also, *the motion of a fluid is "irrotational" when the sum of the circulation round a complete circuit of the boundary of its each small element is zero.*

§ 69. Rotation. Mechanical Illustration.—In order to clearly dissociate the idea of rotation in a fluid from that of circular motion by virtue of which it may possess angular momentum, we may imagine a region of uniform rotation, such as that we have been considering, to have its motion intercepted by a network of rigid boundaries suddenly congealed throughout the region. Then the boundary system will at the instant of its formation receive an impulsive torque, and angular momentum of the rotating mass will be given up, but the rotation within the meshes of the network will persist, the new conditions being those of the supposition in Fig. 37, the equal and opposite circulations along the boundaries in common being materialised. We can suppose a mechanical model constructed to represent this action. Let us imagine a frame mounted upon a shaft capable of revolution, and carrying a multitude of accurately balanced wheels mounted on frictionless bearings, these bearings being arranged parallel, and parallel to those of the main shaft. Let us suppose that the whole apparatus be initially rotating *en bloc;* then if we stop the motion of the frame each of the wheels will continue to spin with the same angular velocity as previously, and nothing that we can do with the frame will alter their rate of spin in the slightest. The frame corresponds to the network boundary system and the wheels with the fluid in the meshes.

§ 70. Irrotational Motion in its Relation to Velocity Potential.—We have above defined irrotational motion as follows:—

The motion of a fluid is irrotational when the sum of the circulation round a complete circuit of the boundary of its each small element is zero.

Assuming this definition, it can be shown that *fluid in irrotational motion has a velocity potential.*

Let (Fig. 38) the cell $a\ b\ c\ d$ be any small element of the fluid in which $a\ b$ and $c\ d$ are lines of flow and $a\ c$ and $b\ d$ are normals thereto. Then since the motion in the line of $a\ c$ and $b\ d$ is *nil*, the circulation round the circuit is the sum of the circulations along $a\ b$ and $c\ d$, and since that motion is irrotational, this quantity is zero.

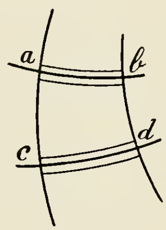

FIG. 38.

Let u_1 be the velocity of the fluid along $a\ b$
u_2 „ „ „ „ „ $c\ d$
x be the distance $a\ b$
x_2 „ „ $c\ d$

(For the sake of simplicity the axis of x has been chosen in the direction of the flow.)

Then let us take two columns of the fluid along the lines $a\ b$ and $c\ d$ respectively, whose section is defined as $\delta y \times \delta z$, then if ρ = density, we have masses of the two columns $x_1\ \rho\ \delta y\ \delta z$ and $x_2\ \rho\ \delta y\ \delta z$ respectively. But their velocities u_1 and u_2 are connected by the relationship $u_1\ x_1 = u_2\ x_2$, or $\frac{u_1}{u_2} = \frac{x_2}{x_1}$. The momenta of the two columns are therefore in the relation $x_2\ x_1\ \rho\ \delta y\ \delta z$ is to $x_1\ x_2\ \rho\ \delta y\ \delta z$, which are equal; consequently, if a certain force applied to any column for a time t will bring it to rest, the same force applied for the same time to the other column will bring that to rest also. But the areas of the columns are equal; therefore to stop or to reproduce the motion of the fluid the pressure difference applied between the points a and b requires to be the same as that between c and d, so that the normals $a\ c$ and $b\ d$ to the field of flow are equipotentials (ϕ = constant).

This demonstration may be taken as applied to every small element of the field, so that the proposition is proved.

Corollary: *When a fluid has velocity potential its motion is irrotational.*

§ 71. Physical Interpretation of Lagrange's ϕ Proposition.—The foregoing proposition, taken in conjunction with that relating to the conservation of rotation, constitutes a demonstration of Lagrange's theorem that "*If a velocity potential exist at any one instant for any finite portion of a perfect fluid in motion under the action of forces which have a potential, then, provided the density of the fluid be either constant or a function of the pressure only, a velocity potential exists for the same portion of the fluid at all instants before or after.*"

This statement, save to a mathematician, is not very clear, as it is difficult to obtain a sufficiently close conception of velocity potential to be able to attach any physical meaning to its conservation.[1] The inversion of the statement, however, obviates all difficulty; it then becomes: *If the motion of any portion of a perfect fluid be irrotational at any instant of time, then, provided the density of the fluid be either constant or a function of the pressure only, the motion of the same portion of the fluid will be irrotational at all instants before and after.*

§ 72. A Case of Vortex Motion.—The case of cyclic motion resulting from an interchange of the functions ψ and ϕ in the source or sink system is one of particular interest. If (Fig. 35) we suppose the origin circumscribed by a line of flow, then we

[1] The velocity potential may fall to zero in a portion of the fluid in the course of its motion without that portion of the fluid losing *the attribute* of velocity potential in the sense of Lagrange's theorem.

have a cyclic system in which the origin represents the axis of a cylindrical body of infinite length making the space round it a doubly connected region. The velocity of the fluid is everywhere inversely as the length of its path of flow, consequently if we suppose the cylinder be made smaller the velocity at its surface will be proportionately greater, so that in the limit if we suppose the cylinder to become evanescent the velocity becomes infinite. The circulation round any such evanescent filament is indeterminate, for it is equal to $\infty \times 0$. The physical signification of this is that we have a system of flow that may be regarded as rotational or irrotational according as we regard the cylinder as non-existent or merely evanescent. If we regard the cylinder as non-existent and the motion as rotational, then the rotation is measured by the circulation round any of the lines of flow (for the circulation round each is the same), so that the whole rotation must be supposed concentrated at the geometric centre.

Such a motion is known as *vortex motion*, and the system figured constitutes a *vortex filament*. It will be seen that if r represent the radius of the path of flow and v the corresponding velocity, $v\,r =$ constant, and if the angular velocity $\omega = v/r$ we have $\omega\,r^2 =$ constant,—that is to say, for any circuit of flow the area $\times$ angular velocity is constant, which is the relation for vortex motion established *generally* by the theorem of Helmholtz and Kelvin. The discussion of this type of motion will be resumed later in the chapter.

§ 73. Irrotational Motion. Fundamental or Elementary Forms. Compounding by Superposition.—All known forms of irrotational motion can be regarded as being compounded from a limited number of different types. These are :—(*a*) Uniform motion of translation ; (*b*) rectilinear motion to or from a point, *i.e.*, sources and sinks ; (*c*) cyclic motion (in multiply connected regions only).

Let us examine first the simple case of a fluid mass possessed only of a uniform motion of translation, and let us suppose that its motion is compounded of two component motions whose velocity and direction are known. Then it is evident that the two component motions can be compounded by drawing a parallelogram, which may either be regarded as a "parallelogram of velocities" if we take its elements to represent velocity, or a "parallelogram of forces" if we take its elements to represent the impulses by which the motion is produced. Thus, if we compound a north wind with an east wind having the same velocity, the result is a north-east wind having a velocity $\sqrt{2}$ times as great; and the forces that would produce the two air currents separately would produce the combined current if acting simultaneously.

If we denote the strength of each superposed stream by a series

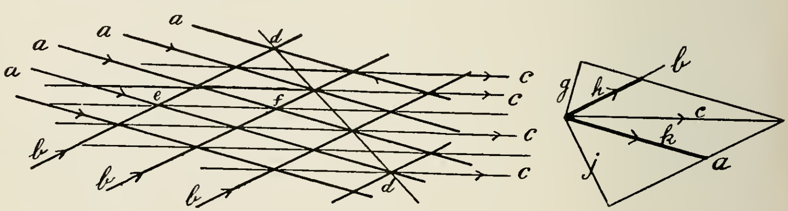

Fig. 39.

of parallel lines, so that the flux or quantity of fluid passed per unit time is the same at every point between each adjacent line and its neighbour—that is to say, if we draw the lines of flow, ψ = constant, for each component stream, then the distance separating any two adjacent lines will be inversely as the velocity,[1] and the network formed by the superposed systems will give the parallelogram of velocities at every point. This method of compounding the two systems of flow is illustrated in Fig. 39, in which *a a a* and *b b b* represent the component streams, and *c c c*, drawn diagonally, gives the resultant flow. It is evident that the lines *c c c* will quantitatively represent equal values of ψ, for the resultant flux across any line *d d* drawn through

[1] Referring to the diagram to the right of Fig. 39, we have from geometrical considerations the normals *g* and *j* respectively proportional to the sides of the parallelogram *h* and *k*.

intersections athwart the stream will be the sum of the components.

§ 74. The Method of Superposed Systems of Flow.—The conception on which the foregoing method has been based can only be applied so long as the fluid moves *en masse*, but it can be shown that the method is applicable to all cases of irrotational motion. If we confine our attention to the field of *force* developed at the instant of application of the component impulses, then it is clear that the resultant field can be obtained by the use of the parallelogram of forces as shown in the figure; there is, however, another, and perhaps more convincing, method of proof; this is the method of superposition.

Let us suppose that instead of two motions being superposed on one fluid current two fluid currents be superposed on one another. This is at first difficult, owing to the instinctive but wholly imaginary difficulty of regarding it as possible for two bodies to occupy the same space at the same time. To simplify ideas, let us suppose the motion to be two-dimensional, so that it may be fully represented on a plane surface; then if we represent one motion on one plane and another motion on a plane adjacent to it the two systems will be superposed; and further, if we take as many systems as we wish and represent them on as many adjacent planes they become superposed. And since a plane possesses no thickness, such superposed systems, however numerous, occupy no finite quantity of the third dimension, and in fact constitute but one plane.

Now, reverting to the argument, let us suppose that any two systems of fluid motion be superposed one on the other. Then so long as we can identify the particles belonging to each separate system (as supposing the streams to consist of different kinds of matter), the two systems must be regarded as separate; but if we imagine that we cannot distinguish the matter in the one stream from that in the other, then a flux across any imaginary barrier in one direction will neutralise an equal flux across the same barrier in the opposite direction, and it will only be possible to recognise the resultant flow; thus, as before, if Fig. 39 represent the two superposed streams by the lines *a a a* and *b b b*, the flux across the imaginary barrier line *e f*, due to the stream *a*, will be equal and opposite to the flux across the same line due to the stream *b*, consequently there is no resultant flux across the line

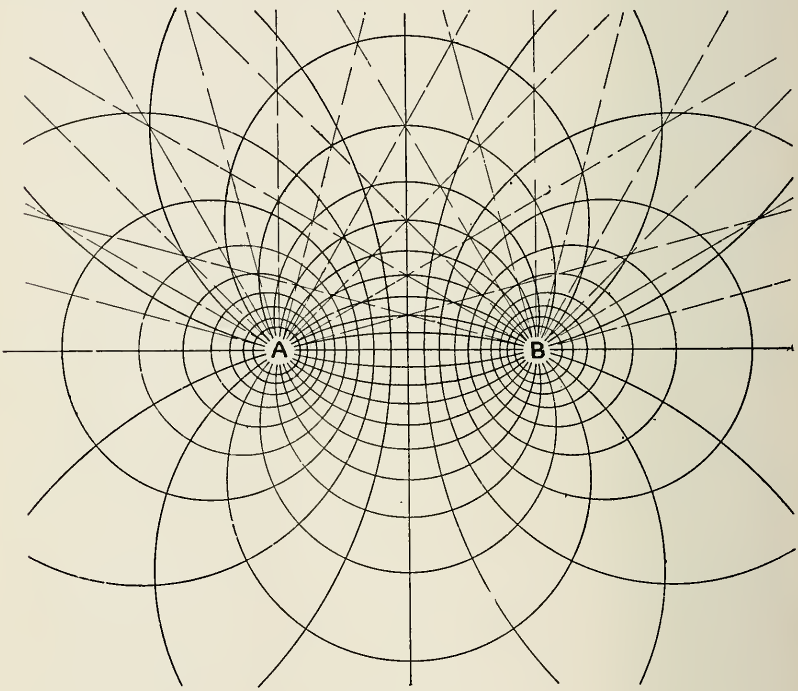

Fig. 40.

e f, which is therefore one of the lines of flow of the resultant system. Likewise in the case of the other parallelograms, so that the field *c c c* is the resultant system.

We therefore see that the superposition of two independent streams has the same resultant as the superposition of two motions on one stream.

The foregoing constitutes the basis of a comprehensive method of plotting the field of flow for any finite combination of known systems. It is the geometrical equivalent of the analytical machinery employed in the mathematical solution of a vast number of cases, *and as such* it is due to Clerk Maxwell. Many compound systems of flow involve an infinite number of elementary components, such as some prescribed distribution of sources and sinks over certain lines and surfaces; the graphic method in such cases is not generally applicable, and the field requires to be plotted from the mathematical solution.

§ 75. **ψ, ϕ Lines for Source and Sink System.**—Let us take the case of a source and sink A and B (Fig. 40), of equal flux, in two

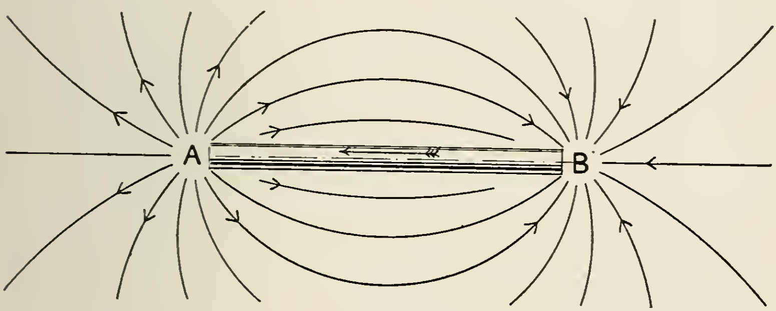

FIG. 41.

dimensions; then the lines ψ constant for the individual fields will consist of equal-spaced radial lines extending indefinitely on all sides, as shown. If now we draw the resultant field we find that the fluid emitted by the source is absorbed by the sink, and from geometrical considerations it is obvious that the paths of flow consist everywhere of arcs of circles passing through the points A and B. Since the functions ψ and ϕ are interchangeable, we can in a similar manner find the resultant system of velocity potential, and we obtain the system of circles shown; if we take the latter as the lines of flow, and the arcs joining A and B as the equipotentials, we have the case of a *vortex pair*, that is to say, two vortex filaments with equal and opposite rotation.

§ 76. Source and Sink, Superposed Translation.—We might (with certain reservations) regard such a combination of source and sink as a tube (Fig. 41) through which fluid is being pumped, the fluid entering the tube at B and emerging again at A. If we suppose such a tube to move longitudinally through the fluid in the same direction as that in which the fluid flows in its interior, or, that which is in reality the same, if we suppose the tube fixed whilst the fluid as a whole has a velocity of translation in the opposite direction, the system of flow undergoes considerable modification.

Fig. 42 gives the solution of such a case for a *two-dimensional field*. The source and sink system of Fig. 40 being superposed on a motion of translation, it is found that two distinct systems of flow result, internal and external respectively to a surface of oval form; the internal system consists of a source and sink in a region bounded externally, and the external system gives the stream lines proper to an oval cylinder in motion through the fluid; it is evident that *we may suppose such a body substituted for the internal system*. The form of this oval represents the shape of a body that will give rise to the same external system of flow as the simple source and sink, and according as the flux of the motion of translation is greater or less in relation to that of the source and sink, the oval will be more or less elongated, the limiting conditions approximating to a line joining the foci on the one hand and to a circle on the other. The form of this oval is not an ellipse, being *fuller* towards the extremities, especially in cases where the ratio of major to minor axis is considerable.

§ 77. Rankine's Water Lines.—These curves and the whole external series have been closely studied by Rankine, the method of plotting here given being that employed by him. Rankine has pointed out the general resemblance of these curves to ships, water lines, and has given them the name "Oogenous Neoids." In a paper read before the Royal Society (November, 1863)

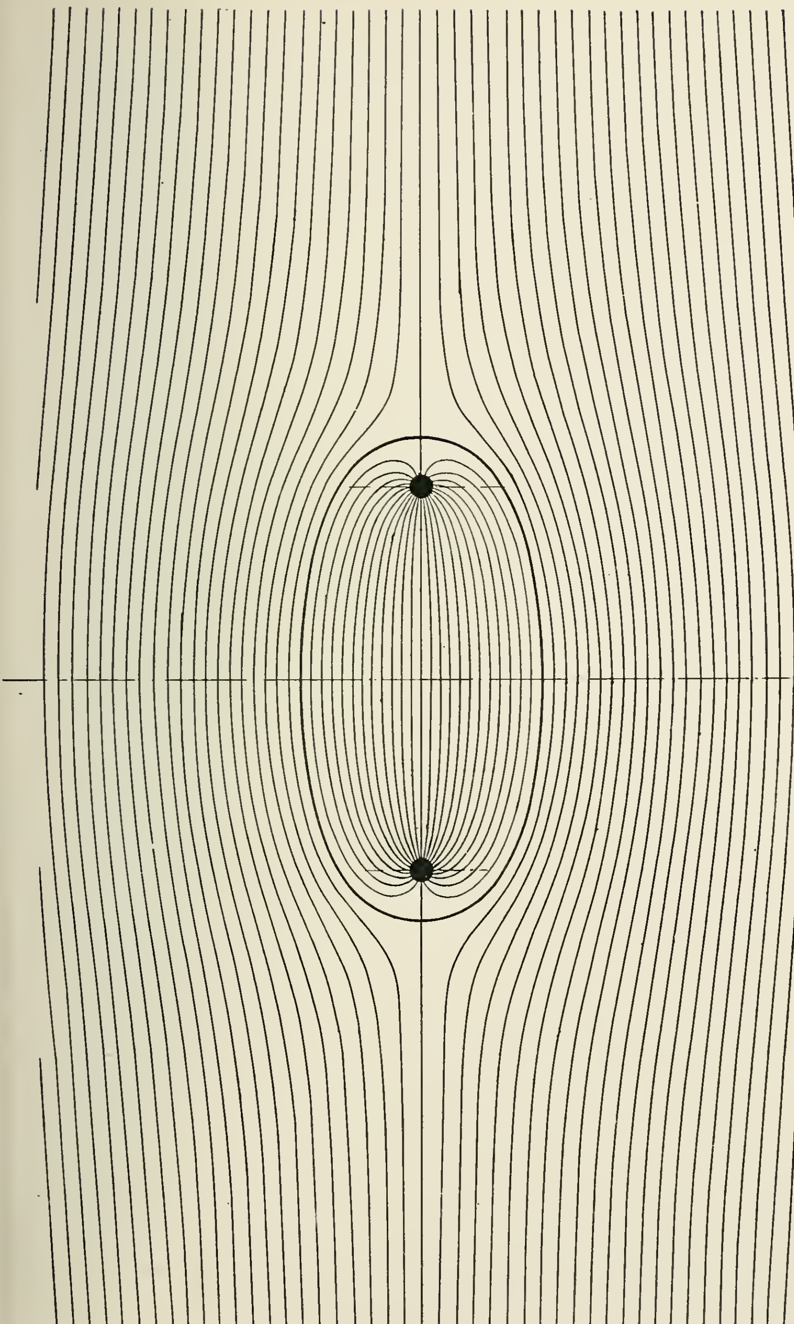

Fig. 42.

Rankine says, referring to the practical employment of these curves :—

"The ovals are figures suitable for vessels of low speed, it being only necessary, in order to make them good water lines, that the vertical disturbance should be small compared with the vessel's draught of water. At higher speeds the sharper water lines more distant from the oval become necessary. The water lines generated by a circle, or 'cyclogenous neoids,' are the 'leanest' for a given proportion of length to breadth ; and as the eccentricity increases the lines become 'fuller.' The lines generated from a very much elongated oval approximate to a straight middle body with more or less sharp ends. In short, there is no form of water line that has been found to answer in practice that cannot be imitated by means of oogenous neoids."

And further :—

"Inasmuch as all the water-line curves of a series, except the primitive oval, are infinitely long, and have asymptotes, there must necessarily be an abrupt change of motion at either end of the limited portion of a curve which is used as a water line in practice, and the question of the effect of such abrupt change or discontinuity of motion is one which at present can be decided by observation and experiment only. Now it appears from observation and experiment that the effect of the discontinuity of motion at the bow and stern of a vessel, which has an entrance and run of ordinary sharpness and not convex, extends to a very thin layer of water only ; and that beyond a short distance from the vessel's side the discontinuity ceases, through some slight modification of the water lines, of which the mathematical theory is not yet adequate to give an exact account."

§ 78. Solids equivalent to Source and Sink Distribution.—In the light of present knowledge it would appear that the particular case of flow under discussion is merely one of an infinite number of possible systems in which sources and sinks of different strengths are distributed along an axis (or *axis plane* for two-dimensional motion in a three-dimensional space), and it is an established proposition[1] that any solid whatever, in motion in a fluid, may be imitated by an appropriate distribution of sources and sinks situated on its surface, and it follows that within certain limitations as to abruptness of contour, an equivalent exists for every stream line solid of revolution in point sources and sinks *distributed along an axis*, and for every cylinder of stream line section in line sources and sinks located on an axial plane. The distribution of sources and sinks that will produce any particular form is only known in a few special cases, such as those of the *elliptical cylinder* and *ellipsoid*, in which the number

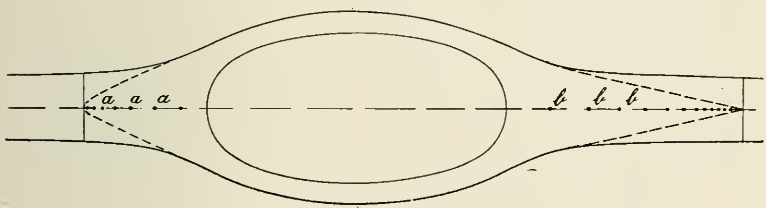

FIG. 43.

is infinite. Any finite distribution can be investigated by the graphic method by repeated compounding of system on system ; a comprehensive way of investigating cases of infinite distribution is at present lacking. It may be noted that in all cases the investigation commences with the source and sink system, the form of the corresponding solid being obtained as a resultant; the reverse process can only be effected by recognising the solid as belonging to some particular system, and consequently only certain solutions are possible.

It is evident that if we take any pair of Rankine's "oogenous neoids" and trim fore and aft to form water lines (Fig. 43), we can regard the process as equivalent to a number of sources in the region *a a a*, and sinks in the region *b b b*, in order to generate and absorb the stream flux that otherwise runs to

[1] Lamb, "Hydrodynamics," pp. 56, 57 (3rd ed.).

infinity in either direction. It would be more proper to discover by trial some combination of sources and sinks that would give an easy termination to the form than to effect this by an arbitrary mutilation, for the true stream lines round the modified form could then be plotted. Beyond this there is no advantage in the one course over the other; the criterion in either case is the eye of the designer. In the hydrodynamic theory of an inviscid fluid, every conceivable body is of stream-line form, and the conditions that obtain in practice do not exist; it is therefore

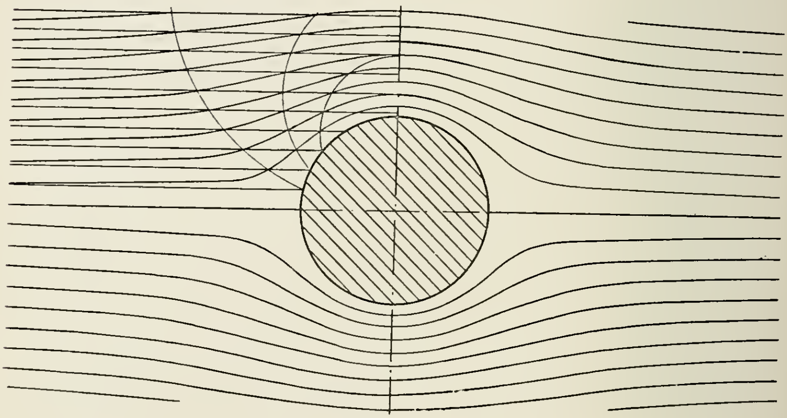

FIG. 44.

useless to attempt to rationalise the ichthyoid or stream-line form by existing analytical theory.[1]

The foregoing example illustrates the graphic method as applied to effecting the combination of motion in two dimensions; certain cases of motion in three dimensions may be solved by

[1] In the paper from which quotations are given it would appear that Prof. Rankine believed there to be some particular virtue in the forms derived from the special case of the simple source and sink system, that the stream lines of such a system constitute in fact *natural* water lines. In actuality ichthyoid or stream-line form is governed by conditions not yet amenable to rigid treatment, and the design of a stream-line form to work in a real fluid with a minimum of resistance is largely a matter of *art*. The underlying principles have been discussed in the previous chapters.

proceeding in the manner laid down by Rankine for a solid of revolution.[1]

§ 79. Typical Cases constituting Solutions to the Equations of Motion.—Some typical cases of stream-line flow constituting the

Fig. 45.

solution to the equations of motion, for the forms of body specified, are given in Figs. 44 (cylinder), 45 (sphere), and 46

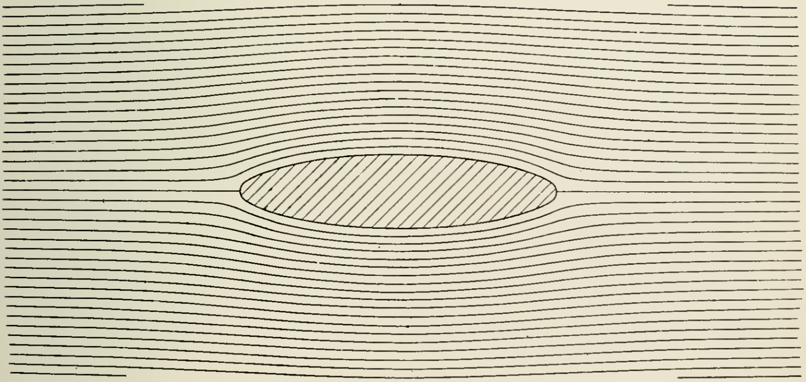

Fig. 46.

(elliptical cylinder). It is scarcely necessary to remark that these forms of flow do not hold good for actual fluids.

In Fig. 47 are plotted the lines of flow for a lamina of infinite

[1] "Principles Relating to Stream Lines," *The Engineer*, October 16, 1868.

extent relatively to the "enclosure," *i.e.*, the fluid at infinity, and relatively to the body itself. In the present work the former are referred to as the *lines of flow* and the latter as the *stream lines*, the latter term being employed in all cases where the primary flow is superposed on a motion of translation. This is merely

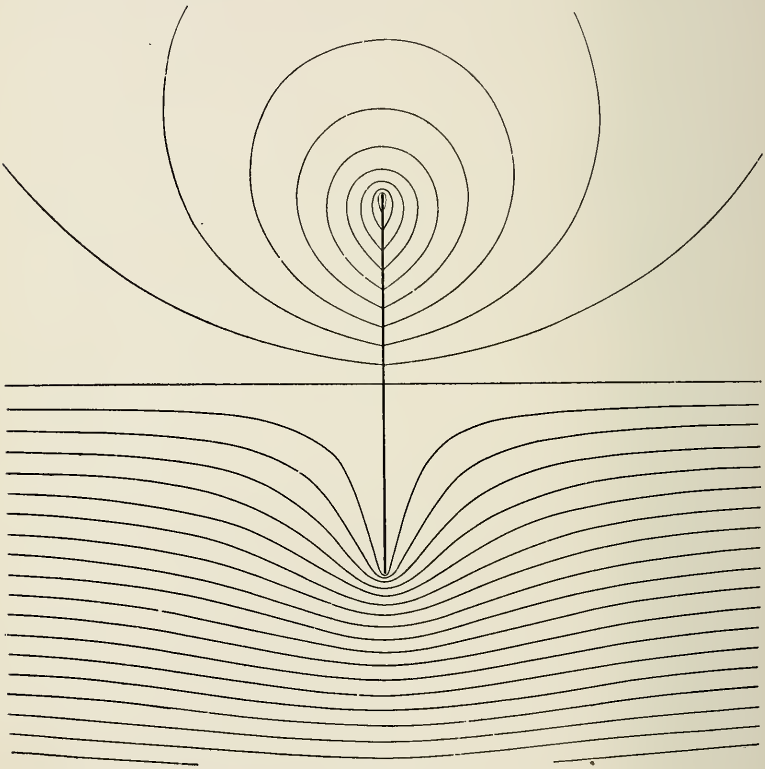

Fig. 47.

a matter of convenience in terminology, in which the present work differs from some of the standard text-books in which the term *stream line* is used more generally.

Of particular interest to the present subject (as will be hereafter demonstrated) is the case of cyclic motion superposed on a motion of translation. Fig. 48 gives the plotting in this

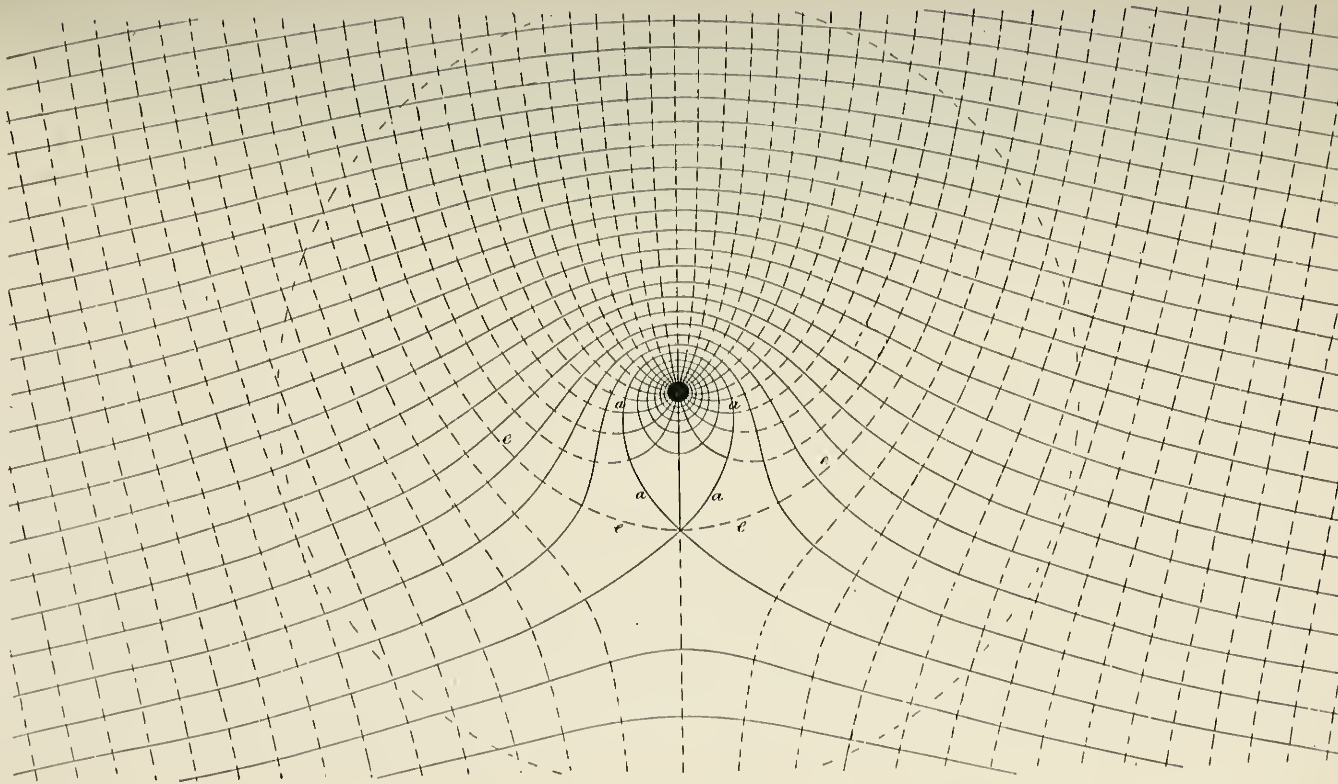

FIG. 48.

case, the cyclic motion being supposed to take place about a filament of negligible diameter; the resultant motion is again found to consist of two distinct systems of flow, one internal and the other external to the surface *a a a a*; the field is plotted in full for equal increments of both ψ and ϕ.

It may be pointed out here that any systems that individually possess velocity potential must of necessity possess velocity potential in their resultant, for otherwise two irrotational systems would, in combination, possess rotation, which is manifestly impossible.

§ 80. Consequences of Inverting ψ, ϕ Functions in Special Case. Force at Right Angles to Motion.—In Fig. 48 the curves of ψ and ϕ if interchanged would obviously give the case of a source or sink, the flow being vertical instead of horizontal. In this inverted reading of the diagram we again find two systems of flow; the surface of separation *e e e e* passes away to infinity, and has parallel asymptotes. It is frequently convenient when reading any flow diagram to temporarily suppose the functions inverted in this way.

A remarkable and important fact in connection with a cyclic system with superposed translation is the existence of a reaction or force at *right angles to the direction of motion*, such force in the case represented in Fig. 48 being an *upward force* acting on the filament, that is to say, a downward force must be applied to the filament in order that the motion as a steady state should be stable. Where the fluid is bounded externally the force must be supposed to act between the external boundary and the filament or such other body as constitutes the inner boundary.

The necessity for this applied force may be demonstrated in several ways, but it is in the first place necessary to consider the distribution of kinetic energy and pressure in the region occupied by the field of flow.

§ 81. Kinetic Energy.—The expression for the kinetic energy of any dynamic system is $\frac{1}{2}mv^2$, where m is the mass and $v =$

velocity. Applying this to the case of stream motion, let b be the distance between stream lines demarcating some definite increment of ψ, then we know that $v \propto \frac{1}{b}$, or $v^2 \propto \frac{1}{b^2}$, that is to say, the energy per unit mass, is, in two-dimensional motion, inversely as the area of the square elements cut off by the ψ, ϕ lines. But the mass of fluid contained in these elements is directly as their area, or varies as b^2, consequently the kinetic energy in each element is proportional to $b^2 \times \frac{1}{b^2}$, which is constant; therefore:

The kinetic energy contained in each element cut off by lines of equal increment of ψ *and* ϕ *is constant.*

In a ψ, ϕ diagram, such as Fig. 48, the total kinetic energy is thus measured by the total number of squares, and the kinetic energy in any circumscribed region is equal to the number of squares in that region. In order to give an absolute value to the energy on this basis it is necessary that the quantity of energy in some particular square element should be known.

The kinetic energy in the field of flow round a body in motion is imparted to the fluid when the body is started from rest, and is given up when the motion is arrested. The effect of the fluid motion is thus to add to the apparent inertia of the body, so that a given force requires to act through a greater distance to impart a given velocity than for the same body *in vacuo*. Not only has a force to act for a greater distance, but also for a *longer time*, which means that the body possesses *in effect* a greater store of momentum for a given velocity. In reality, however, such increase of momentum is only apparent; the momentum of the body and fluid system combined is actually less than that of the body at the same velocity *in vacuo* by the amount due to its fluid displacement. *That is to say, if the body be of the same specific gravity as the fluid the total dynamic system possesses no momentum at all, whatever the velocity.* This apparent paradox is accounted for by the fact that during the period of application of force to the body an equal and opposite force has to be applied to the external boundary of the fluid; thus, if a stream-line body of the same specific gravity as the fluid be started from rest from a boundary surface, during the application of the accelerative force there will be a region of diminished pressure in the neighbourhood, whose sum is in effect of equal value and opposite sign to the applied force. (Compare Chap. I., § 5.)

§ 82. Pressure Distribution. Fluid Tension as Hypothesis.—The distribution of pressure in the field of flow of a fluid in a state of steady motion can be ascertained immediately from the distribution of kinetic energy if we assume the *principle of work.*

The change in the velocity of any element of the fluid in passing from one to another part of the field is due to the difference of pressure on its boundary surfaces, and consequently, on the principle of Torricelli (which follows from the assumption of conservation of mechanical energy), the difference of pressure between any two points is that of the difference of "head" corresponding to the values of the velocity at the two points. Thus if the pressure where the motion is *nil* be taken as zero, the pressure at every point in the field will be proportional to $-(v^2)$.

Now a *minus* pressure constitutes a *tension*, a kind of stress that actual fluids can only support within very narrow limits; we may, however, by subjecting the whole field to a superposed hydrostatic pressure p of sufficient magnitude, do away with *minus* pressure throughout the region, the condition being that for every point $p - nv^2$ is positive, where n is a constant. The pressure under these circumstances becomes, where the motion is *nil*, equal to the applied hydrostatic pressure p.

The objection to the existence of a tension of any desired magnitude in the fluid is entirely based on the behaviour and properties of *real* fluids, with which we are not for the time being concerned; it is a mere matter of hypothesis and definition to provide that the ideal fluid shall support without cavitation any tension whatever, and it leads to some simplification from a physical point of view to make this assumption and deal with the tension system that results. The consequences are the same whether the superposed pressure be taken account of in the ideal fluid, or whether it be regarded merely as a mechanical detail necessary to carrying the theory into the realms of reality.

We already know that the kinetic energy varies everywhere as v^2, and we now have it that the tension also varies everywhere as v^2 (pressure and tension being the same quantity but of reversed sign), consequently the tension on the fluid is everywhere proportional to the kinetic energy, that is the total tension on each element of the ψ, ϕ plotting is constant.

In the interpretation of this and the corresponding result as to energy the two-dimensional diagram must be regarded as consisting of a slice of unit thickness, the energy increment being that contained in the element consisting of the volume cut off by adjacent surfaces, the corresponding tension being measured over the surface of the element.

§ 83. **Application of the Theorem of Energy.**—A simple example of the application of the energy theorem is found in the case of a circular cylinder of infinite length in steady motion in an infinite region containing fluid.

Let Fig. 49 represent the cylinder in cross-section with the external field plotted for ψ and ϕ with respect to space. Let the cylinder be supposed to consist of a thin shell filled with fluid having the necessary motion of translation only; then let the ψ, ϕ lines be plotted for the fluid within the cylinder as shown. Now if we count the complete squares within a quadrant, internal and external to the cylinder, the number is equal; further, for every *part* of a square internal to the surface there is a *corresponding part* external to the surface, and these fractional squares may be made as unimportant as we please by choosing increments of ψ and ϕ small enough, consequently the energy external to the cylinder is equal to the energy internal to the cylinder ; that is to say :—

The energy in two-dimensional motion about a circular cylinder having a motion of translation through a fluid is equal to the energy of motion in the cylinder itself, for equal densities, or the

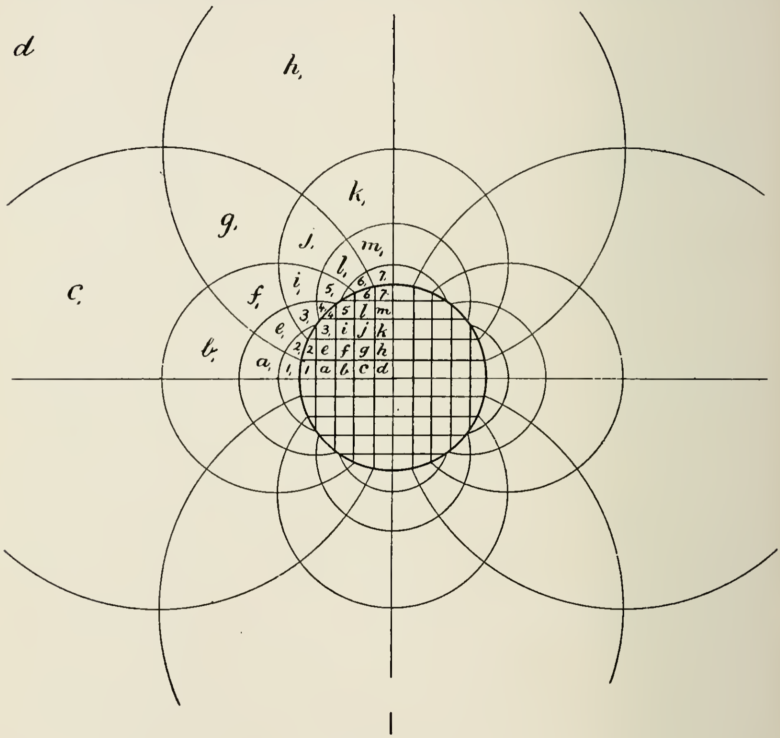

Fig. 49.

energy internal and external are as the respective densities of the cylinder and the surrounding fluid.

In a similar manner it can be shown that the energy accompanying an elliptical cylinder in its motion through a fluid is, densities being equal, the same as for a circular cylinder of diameter equal to the major or minor axis, whichever is placed transversely to the line of motion.

§ 84. Energy of Superposed Systems.—The superposition of systems of flow containing energy may in certain cases result in the addition of their separate energies of motion, but it is evident that this is the exception rather than the rule. The energy of two combined systems is given by the number of ψ, ϕ elements in the combined field.

In the special case, for example, of the superposition of two motions of translation at right angles, as along the axes respectively of x and y, it is found that the energy of the combined field is the sum of the separate energies, a fact which is otherwise obvious (Eucl. 47, I.). In general, it can be shown *that, if on a general motion of translation be superposed any system of flow whose mean velocity in the direction of the translation is zero, the energy of the resultant is the sum of the energies of the component fields.*

Let us suppose the translation to take place along the axis of x, and let the velocity of translation be U; let the x component of the velocity of the superposed system be u a variable in respect of x, y and z. Then the mass of each small element of the fluid is $\rho\ \delta x\ \delta y\ \delta z$, and the energy of the combined field is $\frac{1}{2}\ \Sigma\ \rho\ \delta x\ \delta y\ \delta z\ (U + u)^2$, but $\Sigma\ \rho\ \delta x\ \delta y\ \delta z\ u$ is zero; we therefore have energy $= \frac{1}{2}\ m\ U^2 + \Sigma\ \rho\ \delta x\ \delta y\ \delta z\ u^2$, where m is the total mass; which proves the proposition in respect of motion along the axis of x. But the energy of any components of the superposed motion in the direction of the axes of y and z, which may be regarded as translations at right angles to the main motion, we have already seen also comply. Therefore the total energy is the sum of the components.

§ 85. Example: Cyclic Superposition.—An example may be given in two-dimensional motion in the case of the cyclic superposition (Fig. 48). We know that the energy contained in a case of cyclic motion around a cylinder or cylindrical filament in space is infinite, for the linear size of the ψ, ϕ squares forms a geometrical progression, and any finite number of such squares, however great, may be circumscribed by a circle of finite diameter; that is to say, no finite quantity of energy, however great, will cover the whole field. When the diameter of the filament becomes zero the energy internal to any line of flow also becomes infinite, but so long as we regard the motion as cyclic, we are not entitled to regard the filament as of zero diameter; it is legitimate to suppose the filament of very small diameter, so small as not, *by its size*, to affect the superposed motion of translation.

Now since a pure cyclic motion round a fixed filament does not result in any displacement of the fluid in translation, its mean velocity in each of the co-ordinate directions of space is zero. Consequently, if such a motion be superposed on one of pure translation, the energy of the combined system is the sum of the separate energies and is infinite. Moreover, this result is independent of the energy of the motion of translation (which is never available except to an external system), for if we take the fluid at rest, at infinity (in the x and y directions), and the filament to undergo the translation, the problem is unaffected, and we have proved that to generate a cyclic motion about a filament in motion (Fig. 48) requires the same quantity of energy as to generate the same cyclic motion about a filament at rest, and in both cases where the expanse of fluid is infinite the total energy required is infinite also.

§ 86. Two Opposite Cyclic Motions on Translation.—In the case of the superposition of a system consisting of two cyclic motions of equal value and opposite sign, such as that obtained by the interchange of the functions ψ and ϕ in the source and sink system (Fig. 40), the energy is finite, for the system consists of a limited number of squares, and consequently the energy required to generate such a system about two filaments moving uniformly in space is also finite; the resultant stream lines of such a superposition are given in Fig. 50. Such a system possesses a plane of symmetry $A\ A$, and the motion of the fluid on either side of this plane will be in nowise affected by supposing a rigid boundary substituted for the fluid on the opposite side; Fig. 50 may therefore be read as representing the case of a cyclic motion round a filament in the neighbourhood of a plane boundary, superposed on a translation parallel to the boundary surface, and the energy required to produce such motion is finite.

§ 87. **Numerical Illustration.**—As a numerical illustration and a check on the foregoing, the author has estimated the energy in

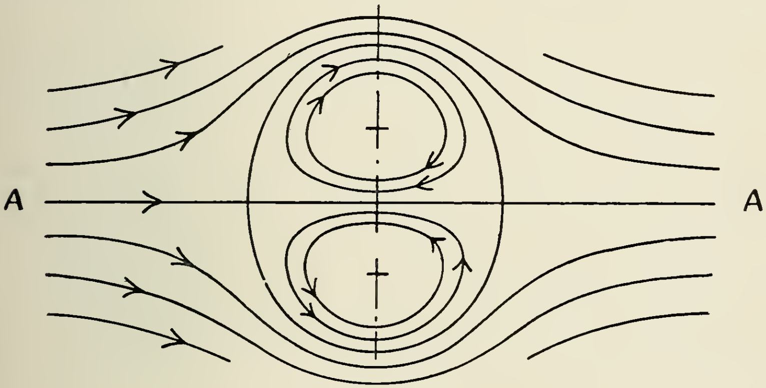

Fig. 50.

the plotting given in Fig. 48, in the region included in the external system within the circular limit indicated, being one of the lines of flow of the cyclic component. The number of squares in the component motions were calculated from the diameter of the circular limit, and the number in the combined system counted, fractions being estimated by a planimeter. The results are as follows:—

Cyclic component	336
Translation	384
Total (sum of above)	720
Total by measurement	719·2
Difference (evidently due to unavoidable error in measurement) . . .	·8

§ 88. Fluid Pressure on a Body in Motion.—The pressure system about a body in motion in a fluid may be regarded as composed of two distinct component systems, *i.e.*, the *accelerative system*, being that developed the instant a force is applied to a body at rest, which is essentially identical with the field of velocity potential, and the *steady motion system*, in which we have seen the fluid experiences a tension everywhere proportional to the energy density (compare "Dynamical Equations," § 59). The first of these is in evidence at the instant when the velocity is *nil*, as when a motion is started from rest or at the instant it is brought to rest; the second system belongs to the steady state when the disturbance is not subject to acceleration. For intermediate states when motion and acceleration are both present the two pressure systems are found compounded. A good illustration is to be found in the case of a body vibrating in a fluid under the influence of a spring, such as a vibrating rod. At the moment such a body is at the end of its motion, when the accelerative force is greatest, the pressure system is that due to the field of velocity potential; when it is in mid-stroke, that is when its velocity is greatest, the pressure system is that of steady state and follows the law already given.

The accelerative pressure system may (as has been already stated) be provided for, so far as the effect on the motion of the body is concerned, by the supposition of an appropriate addition to the mass, and the extent of this addition has already been given in certain typical symmetrical cases on the basis of the energy of the disturbance. When the impulse, as in the case of an oblique plane, is not in the direction of motion, it is not possible to account for the whole effect on the added mass basis, and in fact it is difficult to obtain a clear conception of the physical aspect of the problem in such unsymmetrical cases, and in general the solution is wanting. It would appear in the case of a plane possessing in itself no mass, that the motion on the application of a normal impulse borders on the indeterminate, for a tangential component, however small, would result in an indefinite velocity in an edgewise direction being superposed. It will be seen that in such a case as this the corresponding state of steady motion is unstable, unless a torque be supposed applied from without.

It is established that in the perfect fluid any body in steady motion, no matter what its form, experiences no resistance whatever in the direction of its flight, that is to say, the sum of the longitudinal components of pressure on its posterior surface is equal to the sum of those on the anterior surface. It is only in certain symmetrical cases, however, that the conditions of motion are stable without a force or forces applied to the body.

§ 89. Cases fall into Three Categories.—Taking the body and fluid as a combined system, cases fall naturally into three distinct categories: Firstly, those in which the fluid motion is *in effect* symmetrical, in which case the motion is in equilibrium without any applied force (this includes cases of unstable as well as cases of stable equilibrium). Secondly, cases in which the body is unsymmetrical and in which the motion involves the application of a couple or torque. Thirdly, cases in which cyclic motion is present and in which the motion involves a transverse force. Cases may occur which fall into both categories 2 and 3.

The first category has been sufficiently dealt with already in the present chapter and in Chap. I; the second is typified by the case of the inclined plane, and in a generalised form has been investigated by Kirchhoff, who has pointed out that there are three mutually perpendicular directions for any solid, in which, if it be set in motion and left to itself, the motion will continue indefinitely; in general it has been shown that one only of these directions is stable, the other two represent cases of unstable equilibrium. Generally speaking, a body having an aspect of greatest area such as an oblate spheroid, or a plane disc, tends to move "broadside on," and if its motion at any time is disturbed it will oscillate about such natural "aspect of equilibrium," unless a restraining couple of sufficient magnitude be applied.[1]

The third category possesses a particular interest in relation to aerial flight. The transverse force is characteristic of cyclic motion and is found as a consequence of the superposition of a cyclic motion on a translation, as in Fig. 48. It is due to the greater tension on the upper than the under surface of any circuit, such as that of the solid of substitution, *a a a*; this difference of tension is indicated by the *numerical superiority of the ψ ϕ squares in the region adjacent to the upper surface.*

The connection between cyclic motion and a transverse force can be independently established by taking the transverse force as *hypothesis* and proving cyclic motion as a consequence.

§ 90. Transverse Force Dependent on Cyclic Motion—Proof.—Let A B (Fig. 51) be successive positions of the body or filament at the beginning and end of a short interval of time, to which the transverse force is applied. Let it be granted that the filament exert a force F on the fluid at right angles to its direction of translation, and let us suppose that this force be sustained by a distributed system of forces, f f f_1 f_1, etc., acting from the boundary of the region, and let the line S S represent the mean position of the force F during the period under consideration.

Now the force F forms with the forces f f and f_1 f_1 two couples (which from considerations of symmetry may be taken as equal) of opposite sign, that to the right being counterclockwise and that to the left clockwise. Assuming a steady state, the first of these is continuously engaging with and acting on undisturbed fluid on the right of the line S S, and must therefore be communicating to it counterclockwise angular momentum, and the following couple must be communicating clockwise angular

[1] For a full exposition of the theory of this branch of the subject, reference should be made to Lamb's "Hydrodynamics," Chap. VI., and numerous references therein cited; also "Nat. Phil.," Thomson and Tait, 313, 320.

momentum to the fluid passing into the region to the left of *S S*. But this fluid is the same as that to which counterclockwise momentum had previously been imparted. And the two couples are of equal magnitude, and act on any portion of the fluid for equal time. Consequently the clockwise couple will exactly take away the angular momentum communicated by the counter-clockwise couple, and the final state of fluid will be the same as

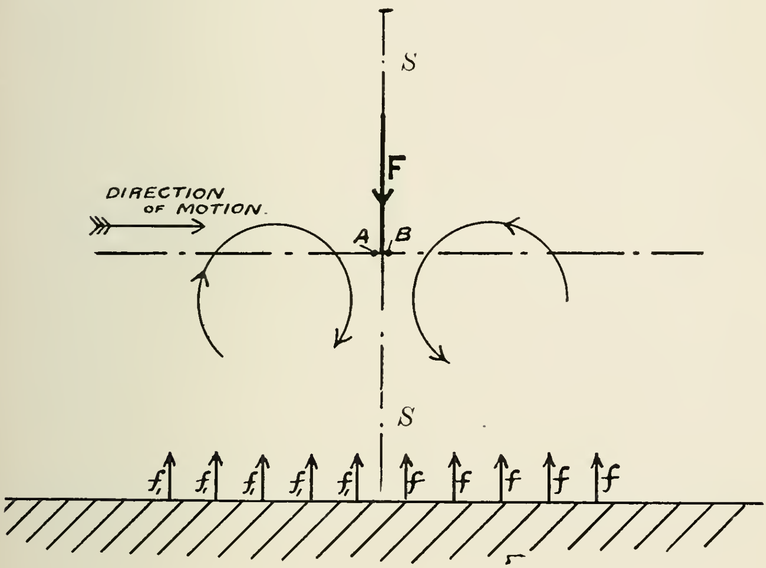

Fig. 51.

its initial state. Also it will possess counterclockwise momentum whilst in the neighbourhood of the applied force. But this implies either a cyclic motion or a rotation, and we know the latter to be impossible; therefore a *transverse force acting between the filament and the fluid implies a cyclic motion around the filament.* It is evident that the foregoing theorem involves as a corollary the converse, *i.e., that a cyclic motion in translation will give a transverse reaction.* We have yet to investigate in what manner, if it is possible, the cyclic motion can be generated.

§ 91. Difficulty in the Case of the Perfect Fluid.—In any actual fluid there can be no difficulty. If, for example, we suppose a plane of infinite lateral breadth gliding edgewise through the fluid to have a force applied at right angles to the direction of motion, this force is borne immediately by the fluid, and the conditions necessary to the development of the cyclic system are fulfilled. In a perfect fluid, however, a plane can move without resistance in any aspect, and thus it is not possible to generate a difference of pressure between its two sides except for the period whilst the normal component of its velocity is undergoing acceleration. Being limited in this manner, the quantity of energy disposable for the production of cyclic motion would appear to be strictly limited, and consequently we may form the following conclusions:—

(1) In an infinite fluid where a cyclic motion, however weak, possesses infinite energy, it will be impossible to generate cyclic motion.

(2) In a finite region it would appear possible that cyclic motion may be induced by a body whose normal motion is accompanied by kinetic energy and which therefore exerts a pressure on the fluid while it is acquiring lateral motion under the influence of the applied force; a portion of the applied force being eventually borne by the cyclic motion developed.

(3) Assuming (2), the more limited the region the less the body will yield to the applied force in the production of the cyclic motion necessary to give rise to an equal and opposite reaction.[1]

§ 92. Superposed Rotation.—If rotational motion be superposed on a motion of translation, equilibrium cannot be maintained by forces applied to the boundary either internal or external.

[1] Conclusions (2) and (3) may be taken as provisional, pending proof or disproof on analytical lines. The inviscid fluid of Eulerian theory is a very peculiar substance on which to employ non-mathematical reasoning. It is quite likely that in the inviscid fluid the dynamic conditions are satisfied without the production of cyclic motion under any circumstances.

Let us take the case of a cylindrical body of fluid rotating *en masse* about its axis. Then we may regard such motion as approximately composed of a number of cyclic motions superposed, and *with their internal boundaries removed.* Let us assume the cylindrical space to be subdivided by a number of concentric cylindrical surfaces, such as the lines of flow of a cyclic system, and, beginning at the centre, let us suppose a cyclic system to be started about a filament so that the velocity at the surface of the filament is that of the rotation. Then, taking the next concentric surface and treating it as a boundary, let us suppose a further cyclic system to be superposed on the first so that the velocity at the surface in question becomes that of the rotation, and again with the next concentric surface, and so on; then by taking the concentric surfaces sufficiently close to one another the motion of the fluid in rotation can be approximated to any desired degree. So long as the boundaries be supposed to exist the system is a superposed series of cyclic motion; if the boundaries be supposed withdrawn the motion is one of uniform rotation.

Now let us suppose such a system superposed on a motion of translation. Each cyclic system will give rise to a transverse resultant force on its boundary so that we shall have forces acting throughout the fluid occupied by the rotation. It is here assumed that the fluid is *constrained* to follow the paths of motion as geometrically laid down as the result of superposition, and it is shown that such constraint involves forces acting from without distributed over the whole region occupied by the rotation, a thing which under the conditions of the hypothesis is impossible of achievement.

The impossibility of compounding rotational motion with translation otherwise follows directly from Lagrange's theorem, for the resultant would involve the transfer of rotation from one part of a fluid to another, and would thus involve the violation of a principle that is fundamental.

§ 93. Vortex Motion.—It is unnecessary in the present work to do more than give a general description of vortex motion and vortices, and discuss their properties so far as bearing on the present subject.

Reference has already been made to *vortex motion* in § 72, where the character of the motion in a vortex filament is dealt with, and it is shown that such a filament possesses *rotation*, and the relation *area* × *angular velocity* = *constant* is established.

The most common form of vortex motion is found in the *vortex ring*, familiar from the easy manner in which such rings can be produced in smoke-laden air (the smoke being necessary to render the rings visible), either by ejecting tobacco smoke from the mouth or by employing a simple apparatus consisting of a box having a circular aperture on one side and a loose diaphragm on the other. Vortex rings of great size may frequently be seen when a salute is being fired from guns of large calibre.

The motion in a vortex ring resembles that of an umbrella ring being rolled on its stick, only the rotation is in the reverse direction—that is, as if the ring were being rolled inside a cylinder; the fluid is, so to speak, being eternally turned inside out, with a motion of translation superposed. The superposed translation is necessary to its equilibrium.

In real fluids the *rotation* is not concentrated at the *axis* as in the case discussed in § 72, but is distributed about the axial region or *core*. As a matter of convention in the perfect fluid, it is usual to suppose the core to be in a state of uniform rotation—that is, to have constant angular velocity, and the motion of the part external to the core to be cyclic and irrotational, there being no discontinuity at the surface of the core, the velocity (u v w) being a continuous function of the position (x y z).

By this convention the core behaves as a solid body, since in an inviscid fluid under no circumstances can its rotation be destroyed or transferred.

We might equally suppose the core to consist of a void space, a region of cavitation in fact, whose pressure is zero. Spiral vortices of this type occur when a screw propeller gives rise to cavitation, the void being filled with water vapour. In other cases the core is constituted by a region filled with some other fluid. We again find an example in the motion produced by a screw propeller when the tips of the propeller emerge from the surface and carry down with them into the water a quantity of air—such vortices may frequently be seen astern of a vessel when steaming under a light load.

A vortex cylinder or filament may be regarded as a portion of a vortex ring of infinite diameter; it can only exist

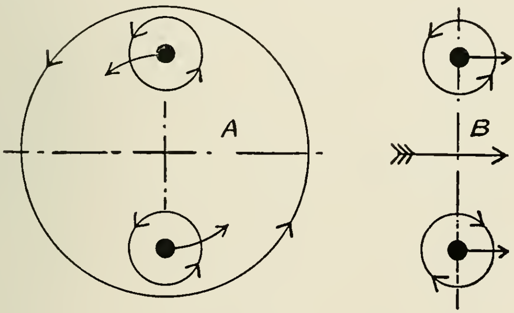

Fig. 52.

either if infinitely long or if its ends terminate on boundary surfaces.

A single straight filament in infinite space is theoretically stable without motion of translation; two such filaments in the neighbourhood of one another mutually interact, and are only stable with superposed motion.

The superposed motion proper to two filaments depends upon their relative position; parallel filaments of like rotation rotate round one another at a velocity proper, each to each, to the cyclic motion of its neighbour, as in Fig. 52 (A); parallel filaments with counter-rotation are in equilibrium when possessed of motion of translation (Fig. 52 (B)); when two such vortices are equal to one another the combination is termed a *vortex pair*, and the direction of the translation is at right angles to the plane containing their centres.

A vortex filament in the neighbourhood of a plane boundary surface behaves as if it can see its own reflection, that is, as if such reflection were another vortex filament.

A vortex ring may be looked upon as the analogue of a vortex pair in three dimensions, *i.e.*, the mutual interaction of its parts results in a motion of translation, the translation taking place at right angles to the plane of the ring.

Two concentric co-axial vortex rings tend to behave as two similarly rotating filaments, *i.e.*, revolve round one another; the consequence of such a motion under the changed conditions is that the two rings alternately overtake and pass through one another, the process being repeated and going on indefinitely. Rings behaving in this way are sometimes said to play "*leap-frog*."

Groups of filaments or rings behave in a similar manner to pairs: thus a group of rings may play "leap-frog" collectively so long as the total number of rings does not exceed a certain maximum; congregations of vortex filaments likewise by their mutual interaction move as part of a concerted system, like waltzers in a ball-room; when the number exceeds a certain maximum the whole system consists of a number of lesser groups.

In general, beyond the special features above described, the motion and behaviour of vortices and vortex rings presents much in common with that of solid bodies; thus two vortex rings on impact bounce off from one another like two perfectly elastic solids, and we have the *Vortex Atom* theory first propounded by Lord Kelvin (Sir William Thomson), and subsequently extended by Professor J. J. Thomson (ref. "Motion of Vortex Rings," Macmillan, 1883).[1]

[1] The present description of vortices and vortex motion is a bare statement of the most elementary facts of the subject. Most of that which is known will be found in the writings of Helmholtz, Kelvin, and J. J. Thomson, and a mathematical *resumé*, with copious references, in Lamb's "Hydrodynamics," Chap. VII.

§ 94. Discontinuous Flow.—Up to this point the assumption has been made that the continuity of the fluid cannot be broken. As a working hypothesis, the fluid has been defined as capable of sustaining stress in tension; it has at the same time been pointed out that the equivalent result may be obtained by supposing the fluid to be subjected from without to a hydrostatic pressure superior to the greatest negative pressure (tension) due to its motion at any point throughout the region.

We will now suppose that the fluid is not capable of sustaining tension, and that the external hydrostatic pressure is either wanting or is insufficient to prevent cavitation.

The importance of studying these conditions does not rest so much upon the possibility of actual cavitation, as upon the general resemblance of the resulting systems of flow to those encountered where real fluids are concerned. It is evident that the void regions in the examples we are about to discuss may be supposed filled either with some different fluid, or even with inert masses of the same fluid as that in which the motion is taking place.

§ 95. Efflux of Liquids.—A typical example of motion with a free surface is presented in the efflux of liquids. When a liquid escapes from an orifice under pressure, the surfaces of the jet so formed, and its interior a short distance away from the point of discharge, are at atmospheric pressure (presuming the experiment is conducted under ordinary conditions), and the velocity can therefore be predicted, knowing the pressure within the vessel. If we suppose the pressure to be applied by a head of liquid in the vessel, then whatever quantity of liquid passes out of the jet disappears from the region of the free surface, so that if we assume the " principle of work," and suppose there to be no loss of energy, the velocity of the jet will be that due to a body falling freely from the height of the column of fluid measured from the point of discharge to the free surface. This is the theorem of Torricelli.

Let the area of the efflux *jet* be A; let s be the "head" of liquid whose density is ρ; let v be the efflux velocity.

Let it be assumed that the pressure within the vessel is everywhere due to the hydrostatic head,—that is to say, let us suppose that the motions of the fluid within the vessel do not affect the pressure on its surfaces.

Then $v = \sqrt{2\,gs}$, and mass of fluid passing out per second $\rho A v = \rho A \sqrt{2\,gs}$, or,

$$\text{Momentum per second} = v \times \rho A \sqrt{2\,gs} = 2\,\rho A gs,$$

which is the reaction on vessel due to the "recoil" of efflux.

But pressure per unit area on wall of vessel at level of aperture $= \rho gs$, or, if a = area of wall of vessel on which pressure is relieved,

$$a = \frac{2\,\rho A gs}{\rho gs} = 2\,A.$$

That is to say, on the above assumption the aperture in the wall of the vessel is twice the area of the resulting jet.

When the aperture is a simple hole in the wall of the vessel (Fig. 53, *A*), the assumption is not strictly accurate, for the pressure in the region surrounding the hole is less than that due to the hydrostatic pressure owing to the converging of the lines of flow, and consequently the actual hole is of less area than that over which the pressure is effectively relieved, and the jet contracts less than the simple theory would indicate.

§ 96. The Borda Nozzle.—The conditions of hypothesis are most nearly conformed to by the *Borda re-entrant nozzle* (Fig. 53, *B*), in which the aperture is furnished with a short tube extending inward. Such an arrangement ensures, as closely as is possible in practice, that the pressure on the walls of the vessel is unaffected by the motion of the fluid. Experimenting with a circular cylindrical nozzle, Borda (1766) obtained the result $a = 1{\cdot}94\,A$, which is in sufficiently close agreement with theory. It is more usual to invert this expression, writing $A = {\cdot}515\,a$.

The complete solution of the path of flow at the free surface has been effected, in the case of the Borda nozzle *in two dimensions*, by Helmholtz, and may be found in Lamb's "Hydrodynamics," where the solution is also given in the case of a simple *two-dimensional* aperture; the calculated coefficient in the latter case is ·611, which does not differ hopelessly from the experimental value, usually taken for two-dimensional flow to be about ·635.

We may evidently suppose the efflux to take place into a vacuous region, or into one filled with air, or even from one vessel containing liquid into another containing the same kind of liquid;

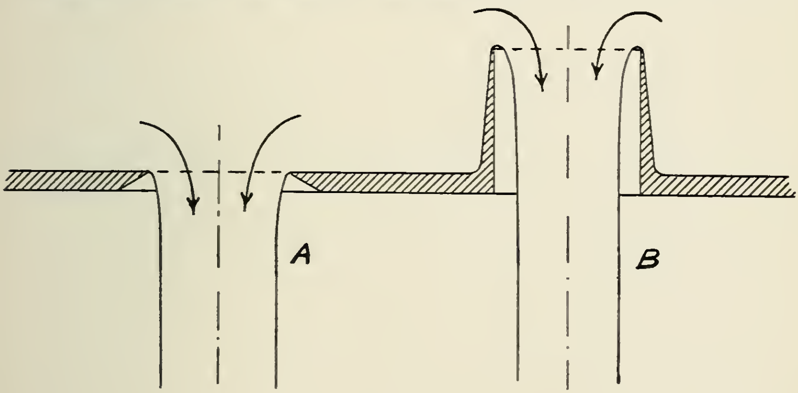

Fig. 53.

the only obvious condition would appear to be that the pressure at all points on the surface of the jet should be constant. Such a system of flow bears a considerable resemblance to that which actually occurs in the case of any real fluid, but on the assumption of continuity it is not the form of flow given by mathematical theory in such a case. If the edges of the aperture are taken to be infinitely sharp, then the discrepancy can easily be accounted for, as the velocity at the sharp edge becomes infinite, and consequently an infinite hydrostatic pressure will be necessary to prevent cavitation, which is not possible; the conditions of hypothesis are therefore departed from. This, however, is not the full explanation, for the flow in practice closely resembles the efflux system, even when the edges are given quite an easy radius. Before discussing this difficulty further another example of motion with a free surface may be given.

§ 97. Discontinuous Flow. Pressure on a Normal Plane.—In Fig. 47 the stream lines are given for a normal plane on the assumption of continuity. We now have to deal with the same example under different conditions, the form of flow involving

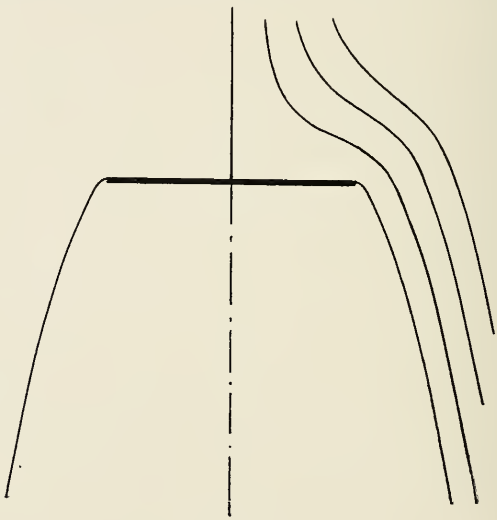

Fig. 54.

discontinuity; a stream of infinite breadth impinges normally on a fixed plane, from the edge of which springs a free surface. The solution to this problem is only known in the particular case of two-dimensional motion where the plane is a lamina of infinite lateral breadth, and is, in the main, due to Kirchhoff. The form of the resulting free surface is given in Fig. 54, in which the direction of flow is taken as vertical.[1] The *pressure force* for one unit width of the lamina is given by the expression

[1] Gravity is assumed to be inoperative. In Fig. 54 the free surface only is an actual plotting; the stream lines are merely an indication of the character of the flow.

$\cdot 44\ \rho\ V^2\ l$, where ρ is density, V = velocity, and l = width of lamina in absolute units. The expression for mean pressure will therefore be :—

$$P = \cdot 44\ \rho\ V^2$$

Similarly the case of an inclined lamina has been investigated by Kirchhoff and Rayleigh, and the following are the expressions obtained :—

$$\frac{P}{\beta} = \frac{\pi \sin \beta}{4 + \pi \sin \beta}\ \rho\ V^2, \qquad (1)$$

where β is the angle of inclination.

For the position of the centre of pressure :—

$$x = \tfrac{3}{4} \times \frac{\cos \beta}{4 + \pi \sin \beta} \times l, \qquad (2)$$

where l is the width of the plane and x the distance forward of the geometric centre.

The following Table gives the result for different values of β (1) pressure on plane in terms of normal pressure, and (2) the proportionate distance of the centre of pressure from the central line :—

β	(1)	(2)
90°	1·000	·000
70°	·965	·037
50°	·854	·075
30°	·641	·117
20°	·481	·139
10°	·273	·163
0°	·000	·187

§ 98. Deficiencies of the Eulerian Theory of the Perfect Fluid.—The deficiencies of hydrodynamic theory have already been pointed out in several instances and partially discussed. The forms of flow that result from the assumption of continuity and the equations of motion bear in general but scant resemblance to those that obtain in practice, and it is not altogether easy to account for the cause of the failure.

If an actual fluid behaved anything like the ideal fluid of theory, the necessity for the ichthyoid form would not exist; any shape, however abrupt, short of producing cavitation, would give rise to stream-line motion and be destitute of resistance. The actual phenomenon of fluid resistance, discussed in the two previous chapters, is characterised by features which at present are not capable of complete elucidation by analytical means.

The principal characteristic in which the actual flow and the Eulerian form differ is as to the existence or otherwise of resistance to motion. In all cases discussed in Chap. I., with the exception of the "stream-line form," the surface or "stratum" of discontinuity is an ever present feature which is closely related to the resistance encountered by the body in motion. It has been shown that the proneness to develop discontinuity *increases* the *less* the viscosity, and it is difficult to understand in what manner the tendency, which grows greater as the value of viscosity approaches to zero, should suddenly cease when zero is reached. This argument may be otherwise stated in the form: It is difficult to understand how a fluid *that offers by hypothesis no resistance to shear* can assume a *rigidity in shear not possessed by a viscous substance.*

§ 99. Deficiencies of Theory (continued). Stokes, Helmholtz.—In the year 1847 Stokes, discussing a particular hypothetical case of flow, was the first to suggest the possibility of a discontinuity or "rift" as a phenomenon connected with the motion of the perfect fluid. Helmholtz, writing in 1868 on the "Discontinuous Movements of Fluids" (*Phil. Mag.*, XLIII.), pointed out the familiar instance of smoke-laden air escaping from an orifice as an example in which the motion is not at all in accordance with the hydrodynamic equation, the air moving in a compact stream instead of spreading out, as the theory of the perfect fluid requires. He remarks that such known facts cause physicists to regard the hydrodynamic equation as a very imperfect approximation to the truth, and that "divers and saltatory irregularities, which everyone who has experimented has observed, can in no wise be accounted for by the continuous and uniform action of [viscous] friction."

This does not express the position of affairs one whit too strongly; in fact, before the date of the recent additions to the mathematical theory relating to discontinuous motion (largely initiated by Helmholtz himself), it might almost have been said that the hydrodynamic theory of the text-book had nothing at all to do with the motions of any known liquid or gas.

In the paper in question Helmholtz states that it is necessary to take account of a condition in the integration of the hydrodynamic equations, which had up till then been neglected. In the hydrodynamic equations, velocity and pressure are treated as continuous functions of the co-ordinates, but in reality there is nothing to prevent in a *true inviscid fluid* two layers slipping past one another with finite velocity. The author of the paper, referring to his previous work on gyratory movement, suggests that the surface of separation is a *gyration surface*, the conception being that the surface consists of an infinite distribution of *lines of gyration* at which the mass of fluid is vanishingly small (or evanescent), and the moment of rotation finite. It is pointed out that such a system involves a discontinuity, such as might be initiated by incipient cavitation, and under these circumstances the conditions of mathematical hypothesis are violated. The theory of discontinuous motions, such as outlined, is afterwards dealt with at some length, with results similar to those already given.

§ 100. The Doctrine of Discontinuity attacked by Kelvin.—The theory of discontinuity has been regarded by some authorities as a questionable innovation, and it has been violently attacked by Lord Kelvin in a series of articles to *Nature* in the year 1894, and so the subject has become a matter of controversy.

So far as the author is aware, this controversy has never been authoritatively settled; it is therefore necessary to give the matter more than passing attention and to discuss the subject from its controversial aspect.

In brief, Kelvin's objections appear to consist in the following: (1) Any system of discontinuous flow is inconsistent with his (Kelvin's) theorem of least energy, and therefore cannot exist. (2) That a surface of discontinuity in an inviscid fluid (whose physical continuity is unbroken) is essentially unstable and, if formed, will break up. (3) That in a real fluid possessed of viscosity a surface of discontinuity is impossible.

§ 101. Kelvin's Objections Discussed.—It is certainly true that the discontinuous system of flow violates Lord Kelvin's theorem; it is evident, however, that this theorem rests definitely upon the hypothesis of continuity, and it is precisely this hypothesis that Helmholtz has deliberately set aside. Consequently the objection is without weight.

In considering the behaviour of an inviscid fluid a certain ambiguity exists. Since *rotation* cannot be imparted to or abstracted from the fluid, there may be an infinite variety of possible forms of flow under given boundary conditions which are ordinarily excluded by hypothesis since they cannot be generated from rest. The Kelvin theorem of least energy is proved only for motions that can be generated from rest, and does not of necessity apply to motions that cannot be so produced.

It is conceivable that if a fluid possessed viscosity in a very small degree only, its motions, if generated and continued for a short period of time, would not sensibly depart from the Eulerian form, but if continued for a long time an entirely different system might eventually be evolved. On this basis, which supposes a *cumulative* change in the form of flow, the inviscid fluid may, after an infinite lapse of time, develop forms of flow quite foreign to the Eulerian theory, and such forms of flow will obviously be independent of Kelvin's theorem. The supposition of an infinite lapse of time merely constitutes an extension of the hypothesis of the perfect fluid, to simulate as far as possible the conditions obtaining in the case of the *nearly inviscid fluid*, discussed further in § 104.

On the second objection, *i.e.*, the supposed instability of the surface of discontinuity in a perfect fluid, we are treading on very different ground, and reference should be made to Kelvin's article. There is certainly nothing to prevent the supposition of the *momentary existence* of a surface of discontinuity in an inviscid fluid, and it is difficult to see how it can be destroyed, in view of the fact that it contains *rotation* which by the theorem of Lagrange can never leave the infinitesimal film of fluid that initially constitutes the surface. It is certain that such a system of flow cannot break up into finite vortex rings, for if the rotation be distributed over a finite quantity of fluid in the core of such vortex rings, the theorem of Lagrange has been violated, and if the rotation be confined to a core that is vanishingly small the energy required to create one such ring is infinite.

§ 102. Discussion on Controversy (continued).—On the third objection, as to discontinuity in the case of the real fluid, it is unnecessary to dwell at length. Neither Helmholtz nor his followers could ever have supposed that the discontinuity exists as a surface under actual conditions, but rather as a stratum containing rotation. It has been elsewhere pointed out (§ 20), that in the case of the real fluid the conception of a surface of discontinuity must be looked upon as an *abstraction of that which is essential* in a somewhat complex phenomenon, and it is this fact that Kelvin appears to overlook; he points out that the surface will, if formed, break up at once into a series of vortex filaments, or vortex rings, and this view is in all probability correct; it may also be found practicable to assess the pressure reduction on the back of a plate on the basis of vortex theory, as suggested in Kelvin's article. It appears, however, to the author that all this may be considered in the light of an *extension* rather than a *controversion* of the Helmholtz theory.

In the course of his criticism Kelvin suggests certain cases of motion as constituting an absolute and patent disproof of the doctrine of discontinuity, which in reality do not seem capable of any such interpretation. One of the supposed cases is given in Fig. 55, which represents a projectile having a gap in its mid-body dividing it into two halves which are assumed to be rigidly connected; this has been indicated in the present reproduction by a stem or spindle.

Now it appears to the author that this example can be construed in favour, rather than otherwise, of the Helmholtz doctrine. Let us suppose the gap bridged initially by a telescopic sheath represented by the dotted line and the projectile set in uniform motion in a perfect fluid. Next let us suppose the sheath to be withdrawn (by sliding it longitudinally), then we have a system of flow involving a surface of discontinuity, a system of

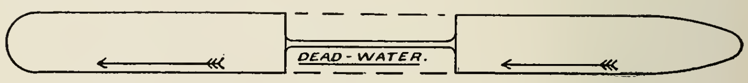

FIG. 55.

flow alternative to that of the ordinary Eulerian theory, and contrary to the theorem of least energy, and *one that has many points in common with that which obtains in practice.*

§ 103. The Position Summarised.—We may summarise the possible causes of the departure from the theoretical Eulerian form of flow as follows:—

(1) The observed departure is due to viscosity, and:

(*a*) *The departure is less the less the viscosity*, as might be readily imagined (to harmonise with the Eulerian theory).

(*b*) *The departure is greater the less the viscosity.*

(2) The departure not necessarily connected with viscosity, and either:

(*c*) *Due to cavitation* (as suggested by Helmholtz).

(*d*) *Due to compressibility* (alternatively suggested by Helmholtz).

(*e*) Due to imperfection of boundary conditions (as suggested by Kelvin).

(*f*) Defect of mathematical hypothesis concerning the nature of an inviscid fluid.

(*g*) The mathematical demonstration in error.

(*h*) The experimental observations in error.

(*j*) Some unaccounted physical conditions.

By a process of exhaustion we dispose of (*g*), (*h*), and (*j*) as highly improbable; (*c*) and (*d*) must be considered as insufficient in view of the fact that no cavitation is in general manifest, and a surface of gyration or discontinuity or vortex motion without an internal boundary involves rotation.[1] Alternative (*e*), suggested by Lord Kelvin, does not seem capable of accounting for the facts known to experiment.[2] It seems evident, under ordinary circumstances, that the boundary conditions are a sufficient approximation to theory.

§ 104. The Author's View.—The true explanation is probably to be sought in (1) (*b*). *In all real fluids the influence of viscosity accounts for the departure;* and the departure is *greater* the *less the viscosity.*

This seems paradoxical; it would appear to denote a sudden change in the behaviour of a fluid when viscosity becomes zero. Such a change would involve discontinuity in the physical properties of a substance, which is scarcely admissible; this paradox is only *apparent*, for the factor of *time* is involved in the production of the discontinuous system of flow, and, as will be subsequently shown, the continuity of behaviour extends to the fluid of zero viscosity.

The following conclusions may be formulated:

(1) That whatever may be the value of the viscosity, the *initial* motion from rest obeys the Eulerian equations.

[1] The compressibility of a fluid does not enable it to evade Lagrange's theorem.

[2] Lord Kelvin in his article suggested the possibility of the boundary conditions being affected by the formation of bubbles at and in the region of sharp corners; but this cannot apply in the case of a gas.

(2) That the discontinuous system may in a viscous fluid be regarded as arising *by evolution* from a motion initially obeying the mathematical equations.

(3) That in fluids possessing different values of kinematic viscosity the time taken for the evolution of the discontinuous system is greater when the kinematic viscosity is less, and *vice versâ*.

(4) That the ultimate development of the discontinuous system of flow is more complete the less the value of the kinematic viscosity, and *vice versâ*.

Taking the propositions in order :—

(1) Forces due to viscosity are proportional to velocity : when velocity is *nil*, such forces have no magnitude, consequently the initial direction of flow is unaffected by viscosity.

(2) In a viscous fluid it is established that the layer adjacent to the surface of a solid is adhesive, *i.e.*, moves as part of the solid—that is to say, the viscous connection between fluid and solid is the same as that between two layers of fluid. Consequently when the flow has been established, there will be a layer of fluid next the solid more or less inert, which will only in a small degree partake of the motion of the dynamic system. Now the surface of the body possesses regions of greater and regions of less pressure, and this inert layer will be steadily pushed along the surface from the regions of greater pressure to those of less. Therefore, taking the typical case of a normal plane, the surface current of fluid so formed will be available to "inflate" the surfaces of hydrodynamic flow in the region of the edges, almost as if the edges of the plane were emitting fluid by volatilisation.

This inflation of the surfaces of flow in regions of least pressure can be conceived to continue until the combined inflated region becomes one whole, the "dead water," occupying the space in the rear of the plane. Similarly for other forms of body.

(3) The less the viscosity the thinner the inert layer, and,

other things being equal, the longer it will take to bring about a given degree of inflation.

(4) The viscous drag experienced between the live fluid and dead water tends to carry the latter away, and if the viscosity exceed a certain value, then, other things being equal, it is found in experience (notably in the case of an ichthyoid form) that the dead water may be ejected and carried away as fast as formed by the viscous drag of the surrounding current. Under these conditions it may be taken that viscosity by its direct drag prevents the surface current from flowing in opposition to the main stream, so that the surface current is consistently rearward, the result being an absence of dead water. The surface of discontinuity may be regarded as having coalesced with the surface film of the body. If the viscosity be sufficiently reduced, the surface of discontinuity will detach itself, and in general *the less the viscosity the more complete will be the development of the discontinuous system of flow.*

Let us now take the case of a fluid bordering on the inviscid. It is evident, firstly, that the change in the system of flow will be very slow; and, secondly, it would appear that the ultimate transformation of the system will be very complete.

Let us now go further and suppose the *viscosity of zero value.* Then, on the principle laid down in § 101, we may regard the ultimate condition as one involving discontinuity as investigated by Helmholtz and others, with the reservation that it will require an infinite time for its development.

The transition stages of the system of flow in the inviscid or nearly inviscid fluid are wholly unknown. If we assume the Eulerian and Helmholtz as the initial and final systems of flow, there must be a continuous series of intermediate stages that await investigation. In the Helmholtz theory the dead water region has assigned to it a pressure equal to that of hydrostatic head. Perhaps the intervening stages could be investigated in like manner by assigning other pressure values to the region in question.

It is by no means certain, however, that the Helmholtz system does actually represent the final form. Since the motion is a matter of infinitely slow development, it is probable that, in spite of the vanishing value of ν, the fluid by which the dead water region is being developed will be set in motion just as in the case of a viscous fluid, the motion taking the form of a vortex ring on a core containing rotation, situated immediately in the wake of the plane or body. Such a system is quite in accord with hydrodynamic principles, but does not involve discontinuity and does not in itself give rise to resistance. It is a pregnant fact that, so long as *the continuity of the system of flow is unimpaired,*

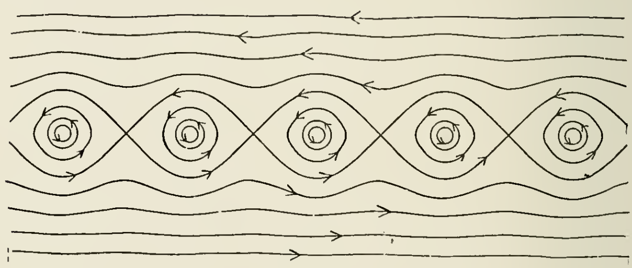

FIG. 56.

the pressure distribution for uniform motion is that of § 88, and resistance other than that directly due to viscosity is absent.

§ 105. Discontinuity in a Viscous Fluid.—It has already been pointed out that the surface of discontinuity in a viscid fluid must begin to degenerate as soon as formed, owing to the fact that a finite velocity between adjacent layers would betoken an infinite tangential stress. We could suppose the degeneration to take the form of a thickening of the discontinuity so that it becomes a stratum of fluid with a velocity gradient. We can alternatively and with every appearance of probability suppose that the surface becomes a stratum of turbulence. The latter would certainly agree more closely with observation.

Suppose we adopt the suggestion of Lord Kelvin and regard the turbulence as initially taking the form of a series of vortex filaments following each other in rapid succession and acting as *rollers* between the live fluid and dead water (on this point Kelvin does not differ materially from Helmholtz), and if we represent the resulting system of flow as in Fig. 56, in which the motion is given diagrammatically relatively to the vortex rollers, so that the apparent motion of the fluid on the two sides is opposite, then, making certain assumptions, we can obtain some results from dimensional theory.

Let it be granted that for different values of V and ν the size of the individual rollers may vary, but the form of the disturbance is homomorphous.

Let ω be the angular velocity of a roller taken at some stated point on some specified line of flow; then,

$$\omega = (F)\ V, \nu.$$

As in § 38, let us write $\omega^x = V^y, \nu^z$.

Dimensionally $$\frac{1}{T^x} = \frac{L^y}{T^y}\ \frac{L^{2z}}{T^z}$$

Thence we have $$x = y + z,$$

and $$y = -2z.$$

Taking $x = 1$, we obtain $y = 2$; $z = -1$.

Hence the expression becomes $\omega = V^2\nu^{-1} \times$ constant, or $V^2 = \omega\nu \times$ constant. Taking r for the radius of the roller at the point chosen, we can write this expression in the form,—

$$r = \frac{\nu}{V} \times \text{constant.}^1$$

§ 106. Conclusions from Dimensional Theory.—From the above expression the following conclusions may be drawn :—

In different fluids *ceteris paribus* the size (diameter) of the rollers will vary directly as the kinematic viscosity. Hence in an inviscid fluid the rollers will become of vanishingly small diameter, or the surface containing them will be a *surface of gyration* of Helmholtz, that is a surface of discontinuity.

[1] This is, as it evidently should be, the same expression as determined generally for homomorphous motion in § 38, r being the linear dimension.

In a fluid of given kinematic viscosity, the size of the rollers will vary inversely as the velocity, that is the velocity difference between the live stream and the dead water.

In a given fluid the *frequency* with which the vortices are generated will vary as the square of the velocity. It is probable that we have in this the origin of the "pitch note" that may be heard when a body is in rapid motion through the air, for example in the swish of a stick or the whistle of a projectile.

The foregoing conclusions are only strictly applicable so long as the vortex rollers are of small diameter compared to the body by which they are generated, for otherwise the motion will not be homomorphous, as required by hypothesis. It is probable that it is the relation between the size of the vortex rollers and that of the body that determines the point at which the discontinuous form of flow begins. Thus for velocities less than a certain minimum in any given fluid the value of r will be so great that there is no room for the vortex to form; at a higher velocity it seems likely that a single vortex may be generated, which will follow in the wake of the body, as in § 104, and it will only be at velocities in excess of this that the vortices will detach themselves in accordance with the *régime* contemplated. The precise conditions must, however, be regarded as uncertain.

The ultimate fate of the vortices formed in the peripheral region of the wake is not altogether known; it would appear that they will break up into groups and sub-groups, after the manner described in § 93, till the whole wake of "dead water" becomes a region of seething turbulence, the motion gradually becoming incoherent and dying out as the energy is absorbed in viscous strain.

CHAPTER IV.

WING FORM AND MOTION IN THE PERIPTERY.[1]

§ 107. Wing Form,—Arched Section.—The most salient characteristics of *wing form* are common to birds of widely different species and habit of life. In spite of variations in detail and in general proportions, there is a certain uniformity of design and construction that cannot fail to impress even the most superficial observer.

The features in common may be taken, on the doctrine of natural selection, as consequent on the form of the fluid motion essential to flight, although, physically speaking, it is the fluid disturbance that depends upon the form of the wing.

To be definite, we may say that the general nature of the fluid motion can be shown to depend upon the major function of the wing, *i.e.*, the support of the weight; the wing form must then conform to the motion so derived, and the detail of the fluid motion in turn will depend upon the more minute character of the wing form. Such indefinite process of adaptation and re-adaptation as the above implies is one to which the methods of evolution appear to be eminently adapted, but to which the methods of calculation are ill suited; hence much of the difficulty of the subject.

One of the most remarkable, and it may almost be said unexpected, peculiarities of wing form is the *dipping front edge or arched section.* This is a characteristic in the wing form of all birds capable of sustained flight, but it is only within comparatively the last few years that this feature has been the subject of observation. It is scarcely credible that so marked a peculiarity should have escaped observation for centuries, but it would seem that such is the case.

[1] Gr. περι and πτερόν (see footnote, Preface).

The wing section given in Fig. 57 is that of a *herring gull* (*Larus argentatus*). The dotted line gives the form as plotted from templates made shortly after the bird had been killed; the full line gives the approximate form in flight when sustaining the weight of the bird. The direction of flight is supposed horizontal.

§ 108. Historical.—Historically, so far as the author has been able to ascertain, the credit of the discovery of the *dipping edge* is due to Horatio Frederick Phillips, whose publication is to be found in the specification of Patent 13,768 of 1884. The discovery

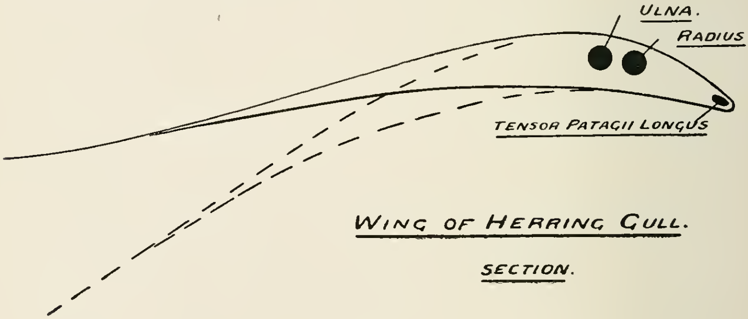

Fig. 57.

appears to have been made as a matter of practical experience, and, as often takes place under these circumstances, the theory given by the inventor in his specification is erroneous. Just, however, as in patent law an inventor's theory, however unsound, is not held to invalidate an invention, so in the matter of *discovery*, the fact that a discoverer does not fully understand the fact that he has been the first to ascertain, does not in any way detract from the credit due. In a case such as the present the fact that the discovery is based on practical experience in the face of an imperfect and in reality hostile theory adds rather than otherwise to its value.

Fig. 58 is a reproduction of the forms of wing section given (as applied to artificial flight) in the specification cited. The motion is supposed to take place from left to right, as in Fig. 57. Fig. 59 illustrates a modified form given in a further specification by the same inventor, 13,311 of 1891. These two figures show clearly the nature of the feature under discussion. In both specifications the theory given is inadequate.[1]

The advantages of the arched form of wing section were known to the late Herr Lilienthal at the time of his experiments in flight, 1890–94, and the discovery has been attributed to him

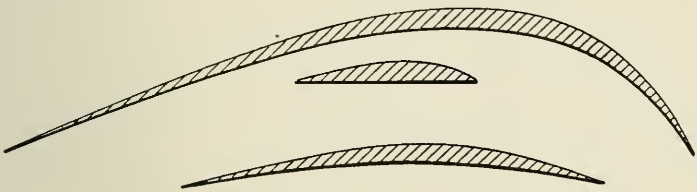

Fig. 58.

by some writers.[2] It is possible that Lilienthal was unaware of Phillips' previous work, and that discovery by him was made

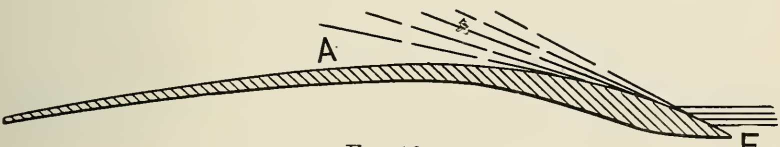

Fig. 59.

independently. There is no evidence to show that Lilienthal possessed more than a practical acquaintance with the arched form.

[1] In his 1884 specification Mr. Phillips says :—

" . . . so arranged that a current of air striking the forward edge of the blade at an acute angle is deflected upwards by the forward part of the surface, and a vacuum (or partial vacuum) is formed on the after-surface, substantially as described."

Further, in the patent of 1891 he writes :—

"The particles of air struck by the convex upper surface A at the point E" (compare Fig. 59) "are deflected upwards, as indicated by the dotted lines, thereby causing a partial vacuum over the greater portion of the upper surface."

[2] See article "Aeronautics," "Encycl. Brit.," O. Chanute.

About the same time as Lilienthal was at work the author succeeded in evolving the arched form, or dipping front edge, purely from theoretical considerations, at that time having no knowledge of the previous work of Phillips or the experiments then being conducted by Lilienthal. The author first formulated his theory in 1892, the basis being the study of the special case of an aeroplane of infinite lateral breadth. Sections of the aerofoil employed in model experiments in 1894 are given in Fig. 60.

The author gave a *résumé* of his theory in a paper read at the

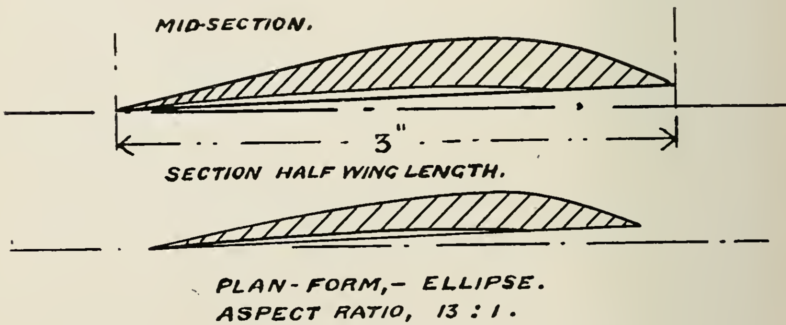

Fig. 60.

annual meeting of the Birmingham Natural History and Philosophical Society on June 19th, 1894, a wall diagram of which Fig. 68 is a reproduction being exhibited. A more complete account of this work formed the subject-matter of a paper offered to the Physical Society of London, but rejected (September 3rd, 1897).

In the present chapter, on wing form and the motion of the fluid in its vicinity, the main argument and demonstration are taken without substantial alteration from the rejected paper, the subsequent work being a revision of the theory on more orthodox hydrodynamic lines.

§ 109. Dynamic Support.—Endeavours have been made in the past to apply the principles of the conservation of momentum—that is, the doctrine of the continuous communication of momentum (§ 3)—to estimate directly the efficiency of an aeroplane sustaining a load and the expenditure of power necessary. If the air were a fluid discontinuous after the manner of the Newtonian medium, then such methods would lead to immediate and reliable results, for we know that if W be the weight supported, and v the velocity of downward discharge, and m the mass per second of the projected particles,

$$W = mv \text{ or } v = W/m \qquad (1)$$

and if E = energy expended per second,

$$E = \tfrac{1}{2}\, mv^2 = \frac{W^2}{2\,m}$$

or for any given weight to be sustained (W = constant) the energy is inversely as the mass of fluid dealt with per second.

Something by way of convention is necessary to connect the above quantities with the size and velocity of the wing member. Thus if we suppose the latter to be an *elastic* aeroplane of area A and angle β, travelling at a velocity V, we shall have :—

$v = V \times 2\, sin\,\beta$ (where v is the velocity imparted *at right angles* to the plane), and $m = VA\,\rho\, sin\,\beta$, and (1) becomes

$$W = \rho\, A\, V^2\, 2\, sin^2\,\beta.$$

If we had taken for our convention that the surface of the aeroplane is *inelastic*,[1] then, since the particles on impact would not bounce off, $v = V\, sin\,\beta$ and

$$W = \rho\, A\, V^2\, sin^2\,\beta. \qquad (2)$$

The above results are not altogether in harmony with experience. The weight sustained does vary approximately with the area of the plane and density of the fluid, and as the square of the velocity, but the relationship in respect of angle does not hold good.

Let us introduce an elementary notion of continuity into the fluid. It is evident that when the layers of air adjacent to the

[1] Compare "Principia," prop. xxx., Book II.

aeroplane are diverted these will react on the neighbouring layers of air, and so on, so that a stratum of some considerable thickness will be involved. Now the factor that must limit the thickness of this stratum is evidently the size and shape of the plane, for the more remote layers of the fluid only escape by the fact that a circulation takes place from the side of greatest to the side of least pressure, which circulation depends chiefly upon the size and shape, and but little upon the angle of the plane. The elasticity of the air might become sensible if the velocity were sufficient, but at ordinary velocities this factor is unimportant.

Let us then assume for our convention that the depth of the layer affected for a plane of given shape depends upon its linear dimension and is constant in respect of angle, the latter being supposed to be of small magnitude. Then, since under the present supposition the lines of flow will require to follow the surfaces of the plane (the fluid being unable to bounce off as in the previous case), we have

$$v = V \sin \beta, \quad m = \rho \kappa A V,$$

where κ is a constant, and by (1) we obtain :—

$$W = \rho \kappa A V^2 \sin \beta. \qquad (3)$$

This result for *planes* of certain general proportions, at small angles to the line of flight, agrees closely with experiment.

The quantity κA of equation (3) may be aptly termed the *sweep* of the aeroplane or wing. It is a measure of the effective cross-section of the horizontal column of air dealt with by the aeroplane or supporting member. It has been found, experimenting with superposed planes,[1] that two planes fifteen inches by four inches in pterygoid aspect, and at angles less than ten degrees, do not suffer any sensible diminution of their individual sustaining power if they are separated by a vertical distance of four inches. It is therefore fair to assume that a plane of the dimensions stated is sustained by the inertia of a layer of air not more than four inches thick. That is to say, the *sweep* does not

[1] Langley.

exceed the area of the plane itself, or we have κ equal to or less than unity.

By employing this in conjunction with equation (3) an outside estimate may be made of the load supported by planes of the form stated. Such estimates generally fall short of the experimental value in the relation of about one to two. By substituting a fictitious value for the "sweep" of about twice that ascertained by experiment the results of the equation can be made to agree. It is evident, therefore, that all the conditions of the problem have not so far been included in the theory.[1]

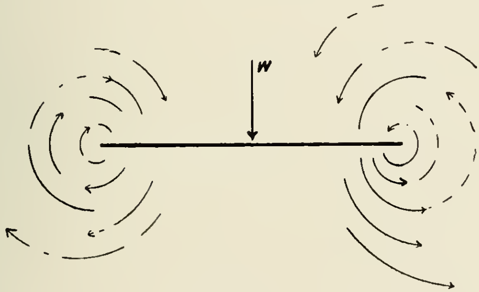

FIG. 61.

§ 110. In the Region of a Falling Plane,—Up-current.—In the foregoing discussion the subject has been treated as if the air, coming into the immediate region of an advancing aeroplane, is in a state of rest, and as if the support is wholly derived from the downward velocity imparted to it. But it has been shown that if this were actually the case the weight supported could, as a maximum, be only about one-half of that found by experiment.

Let us take the simple case of a horizontal plane supporting a weight and allowed to fall vertically. There is *at first* a circulation of air round the edge of the plane from the under to the upper side, forming a kind of *vortex fringe* (Fig. 61), the air all round the edges of the plane being in a state of rapid upward motion.

[1] Compare §§ 160-1.

If now we impress upon the plane a simultaneous horizontal motion, it is evident that the air encountered by its leading edge will be in a state of upward motion, and it would appear probable that this up-current, in front of the advancing plane, would only cease to exist when the horizontal velocity of the plane becomes equal to the velocity of sound.

But if an up-current is encountered, impinging on the advancing edge of a loaded aeroplane, the downward momentum communicated to the air will be augmented, and may be regarded as consisting of two parts, to the *sum* of which the sustaining force is due, *i.e.*, the part communicated in bringing the up-current to a state of rest and the part communicated to the air as velocity downwards.

It is evident that the problem as above presented is in effect identical with that of an inclined plane moving horizontally—that is to say, the relative direction of the horizon is not of importance. The force of gravity in the one case can be substituted by the resultant of the force of gravity and an applied force of propulsion in the other.

§ 111. Dynamic Support Reconsidered.—When we consider part of the support of a body as derived from an *up-current*, it is necessary to examine the origin of the up-current, for it is evident that the generation of such a current must give rise to a downward reaction, and everything depends upon whether such reaction is borne by the body itself or by the deeper layers of the air, and eventually by the earth's surface.

Reverting to the case of a body supported by the communication of momentum to a number of independent material particles, it is evident that the particles projected downwards eventually give up their momentum on striking the surface of the earth. We may follow the subsequent history of the particles in two extreme cases :—

Case 1.—If the particles or the earth's surface are supposed quite *inelastic*, the impact is accompanied by a continual loss of energy, which is given by the expression $\frac{m v^2}{2}$ foot poundals per second.

Case 2.—If, on the other hand, the particles and the surface of the earth be perfectly elastic, the former will rebound with a velocity equal to that with which they strike, and the system as a whole will not lose energy. If the body be arranged to deal continually with the same set of particles, none being allowed to escape, then it may be supported without any continued expenditure of energy—that is to say, without any work being done. Such a case is exemplified in the dynamical theory of heat when a loaded piston is supported by gaseous pressure in a closed cylinder. We could also suppose it to be effected by imbuing the supported body with sufficient intelligence and skill so to direct the particles that they would always rebound within its reach.

We have already seen (§ 4) that in Case 1 the weight supported is equal in absolute units to $m v$. But in Case 2 the particles impinging on the body impart as much momentum as they do in leaving it; hence the *supporting force* = $2 m v$.

In both cases it will be observed that the projected particles act as carriers of momentum between the earth's surface and the dynamically supported body, the weight of which is eventually carried down and distributed on the surface beneath; and, moreover, we are unable to conceive of any arrangement of material particles used for dynamic support, however complex, that will not eventually transmit the stress produced by the weight of the body down to the surface of the earth. (Compare § 6.)

§ 112. Aerodynamic Support.—We may now examine and discuss the behaviour of an incompressible and frictionless (inviscid) atmosphere with respect to an *aerofoil*[1] traversing it.

When a loaded aerofoil is dynamically supported by a fluid, we know that its weight is eventually sustained by the surface

[1] From Greek ἀέρος and φυλλον (lit. an *air leaf*). Compare § 128.

of the earth, and that the transmission of the stress is effected by the communication of momentum from part to part, and is thereby distributed over a considerable area as a region of increased pressure. But, as is usual in fluid dynamics, there is a certain ambiguity in the application of the principle of the continuous communication of momentum, and we as yet lack some definite statement as to the application of the principle to the case in point.

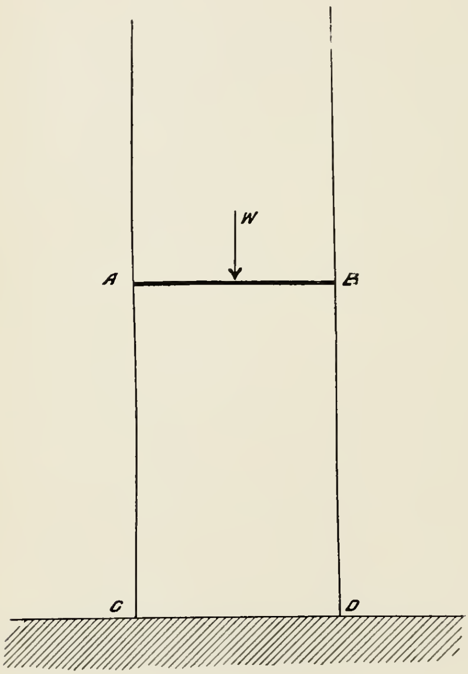

Fig. 62.

In Fig. 62 $A\ B$ represents an aerofoil, supporting weight, W, dynamically, under the conditions of the hypothesis. Consider a *fluid prismatic column* formed by imaginary vertical surfaces touching the edges of the aerofoil and continued downwards to the earth's surface and upwards indefinitely. Adopting the hypothesis that the fluid is inviscid, all forces acting on the column from the surrounding fluid must be normal to its surface, and have no vertical component. The only vertical forces acting on the column are therefore the weight of the loaded aerofoil, W, acting downwards, and the pressure on the base of the column $C\ D$, due to the distribution of the weight W on the earth's surface. Let this latter equal w; there is then a downward resultant $W - w$ acting on the column. (The weight of the column itself and the pressure produced thereby on $C\ D$ are obviously in equilibrium, and require no consideration.)

When the aerofoil has a horizontal motion through the fluid the conception of the prismatic column will not thereby be altered. Although its contents are constantly passing out on one side and being renewed on the other, the instantaneous condition of the forces acting is not in any way affected; the downward static resultant $W-w$ remains. Consequently the downward momentum imparted per second to the fluid leaving the prism *plus* the upward momentum received per second from that entering must be equal to $W-w$.

When the height at which the aerofoil is sustained is great in comparison with its own dimensions, the area over which the weight is distributed on the earth's surface is obviously also great, and the quantity w becomes negligible. Under ordinary conditions this would usually be the case, so that the weight may be regarded as in no part statically supported. In special cases, however, w may become of sensible magnitude, and it is probable that results obtained with a very large aeroplane near the surface of the earth would be found not to hold good for the same aeroplane at any considerable altitude.

§ 113. Aerodynamic Support,—Field of Force.—We have already (§ 60) learnt to regard the lines of flow of hydrodynamic theory in the light of "lines of force" and the region occupied by such lines as a "field of force." The definition may be given as follows:—

A line of force in a fluid is defined as *a line lying everywhere in the direction in which the particles of the fluid are undergoing acceleration*, and in the case of a fluid initially at rest at the instant of its being set in motion the *lines of force are identical with the lines of flow of mathematical theory*. The whole region occupied by the lines of force is termed a *field of force* whose intensity is everywhere proportional to the acceleration of the particles.

In the case of the field proper to a force of stated direction applied to a given body in a quiescent fluid, it follows from considerations belonging to hydrodynamic theory that the form of the field is unique, that is to say, its geometry is absolutely defined by the conditions.

In the case of a fluid in an arbitrary state of disturbance, the field of force will *not* generally be of the same form as for the quiescent state. Where there is pre-existing motion in the fluid we may speak of the field as a *distorted field*.

The form of the field in the case of a fluid initially at rest for such forms as a sphere, an ellipsoid, or a circular or elliptical cylinder, is perfectly well known (§ 79), and in an infinite expanse of fluid extends indefinitely in every direction. If, however, the region is bounded as in the case of the atmosphere, limited by a rigid boundary constituted by the surface of the earth, the field will be modified as represented diagrammatically in Fig. 63, in which the continuous lines are the lines of force, and the dotted lines, normal to the former, are lines or surfaces of equal pressure.[1]

It is a necessary consequence of the definition of lines of force that all lines in the immediate vicinity of a stationary boundary surface must be parallel to it, and therefore that surfaces of equal pressure, if they meet the ground, must do so normally, as indicated in Fig. 63. This figure will consequently represent diagrammatically the spreading out of the pressure area and its ultimate distribution as a region of increased pressure on the surface of the earth.

§ 114. Flight with an Evanescent Load.—We will now suppose that the aerofoil that gives rise to the field of force is *in flight*, that is to say, it possesses a horizontal velocity. Now we know at present very little of the nature of the disturbance created. We cannot even assert that the form of the resulting flow is

[1] Compare § 60. Lines of *equal pressure* only for initial motion otherwise correspond to $\phi_{\cdot}'' =$ const. of mathematical theory.

geometrically similar for different load values; in fact, it will be hereafter shown that it is not.

We will in the first instance direct our attention to the case where the load is supposed to be very small indeed, so small in fact as not to be measurable in finite units, *a small quantity of the second order.*

In order that there shall be no ambiguity in respect of the

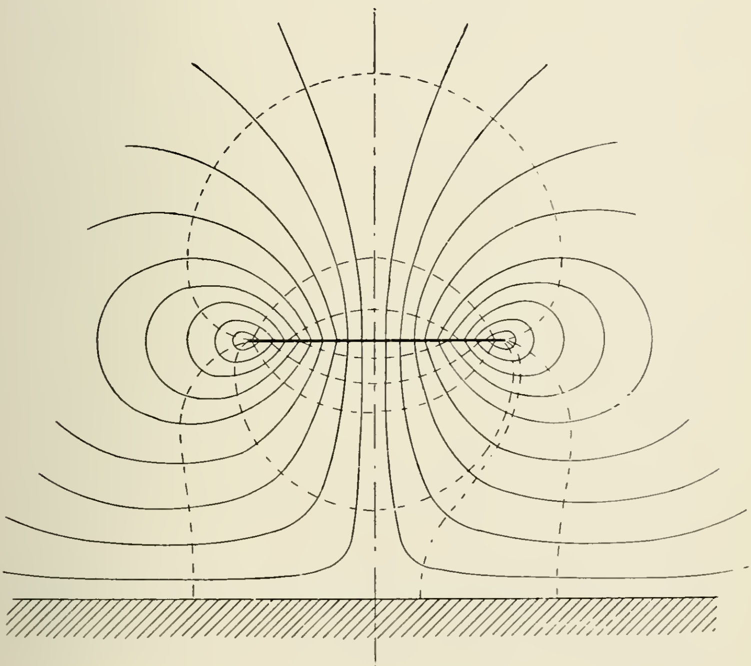

Fig. 63.

proposed conditions, let us imagine a number of aerofoils of equal area carrying different loads that vary from some finite value down to zero, and suppose that each aerofoil is of the best form possible for deriving the support necessary from the atmosphere. Then the form of the aerial disturbance may vary in the different cases, but as the load approaches zero the aerofoil approximates more and more closely to an aeroplane, and the disturbance approximates to its evanescent form, that which we now propose to investigate. We therefore base the initial argument upon the case of *an aeroplane gliding horizontally and edgewise, supporting a load smaller than can be specified in finite units.*

§ 115. Aeroplane of Infinite Lateral Extent.—We have in the previous chapter become familiar with the simplification that results from the consideration of cases in which the motion takes place in two dimensions only, and with the conception of

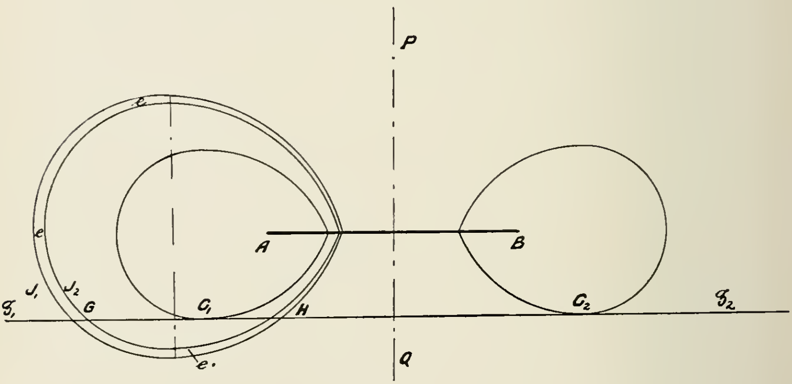

FIG. 64.

bodies of infinite lateral extent as a special case involving such a condition.

In Fig. 64, let A represent the forward and B the after-edge of an aeroplane extending to infinity in the direction at right angles to the plane of the paper; or, if preferred, we may consider the plane to be of finite extent, but bounded laterally by two continuous parallel walls rising vertically from the surface of the earth.

Let us examine a portion of the field $e\ e\ e$ enclosed between two adjacent lines of force, J_1, J_2. *Then the intensity of the field in the region $e\ e\ e$ is inversely proportional to the distance between the bounding lines of force.* For let q_1, q_2, be the normal distances at any two points G and H, and let us suppose a small displacement to take place in the direction of the lines of force. Let this displacement at G and H be equal to s_1 and s_2 respectively; then, since the flux is everywhere equal, $\frac{s_1}{s_2} = \frac{q_2}{q_1}$.

But the acceleration of the particles is proportional to the rate of displacement, and therefore to the displacement itself. Hence $\frac{Acceleration\ at\ G}{Acceleration\ at\ H} = \frac{s_1}{s_2} = \frac{q_2}{q_1}$, that is, the intensity of the

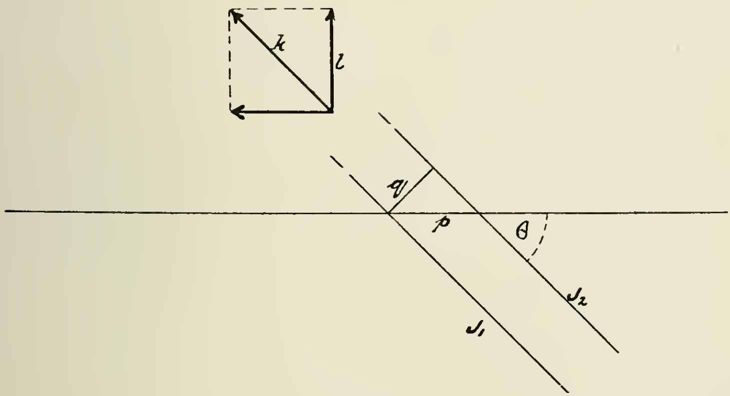

FIG. 65.

field is inversely proportional to the distance between the boundary lines of force.

Taking the velocity of the fluid through the field as V, let k be the intensity of the field (Figs. 64 and 65), where q is the normal distance between two adjacent lines of force, J_1, J_2 (so that kq is constant), and let p be the distance in the line of relative motion, and θ be the angle at which the path of the particle cuts the lines of force. Then the time taken by the particle to traverse the "tube of force" J_1 J_2, $= \frac{p}{V}$, the momentum imparted in the direction of the lines of force $= \frac{k\,p}{V}$, of which the vertical component l is:—

$$\frac{k\,p\,sin\,\theta}{V}$$

But

$$p = q \operatorname{cosec} \theta$$

$$\therefore \quad l = \frac{k\,q \operatorname{cosec} \theta \sin \theta}{V} = \frac{k\,q}{V},$$

which is constant. Therefore, *if the particle, after cutting the tube at G (Fig. 64) and continuing its course, recut the same tube at H, the upward momentum communicated at G will be equal to the downward momentum communicated at H.*

But a particle of fluid traversing the field of force of the aeroplane may be regarded as passing through a series of regions bounded by adjacent lines of force, to each of which the foregoing result may be applied. Consequently the upward

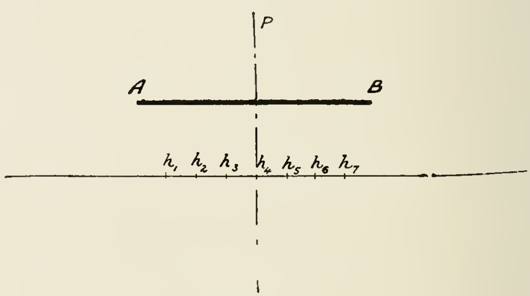

FIG. 66.

velocity acquired in traversing the ascending field to C_1 will be given up in traversing the descending field to the medial line $P\,Q$ (the line separating the front and rear portions of the field), and the downward velocity imparted to the particle in cutting the descending field to C_2 will be given up in traversing the corresponding ascending field, so that, in respect of the vertical component of motion, the final state of the fluid will be the same as its initial state.

Again, since the conditions determining the form of the field are symmetrical, the field itself must also be symmetrical about the plane of which the medial line $P\,Q$ (Figs. 64 and 66) is the *trace*.

Let h_1, h_2, h_3, h_4, h_5, etc. (Fig. 66), be points on the path of a particle of fluid cutting $P\ Q$ at h_4, corresponding to equal intervals of time. In the elements $h_3 h_4$ and $h_4 h_5$ the horizontal components of the forces acting on the particle are equal and opposite; therefore the loss of horizontal velocity along $h_3 h_4$ is equal to the gain along $h_4 h_5$, and the horizontal velocity at h_3 is equal to that at h_5. Similarly the horizontal velocities at h_2 and h_6 are equal, etc., and in general the horizontal velocity component of any particle on one side of $P Q$ is equal to that of the similarly situated particle on the other side. But the original state of the fluid is one of no horizontal motion.[1] This, therefore, is also the final state.

We have consequently shown, in a system such as we have established by the present hypothesis, that the motion imparted to the fluid is eventually given up by the fluid both in respect of its vertical and horizontal components, and consequently there is no continual transmission of energy to the fluid, and no work requires to be done to maintain the motion or to support the plane. The fluid in the vicinity of the aeroplane is in a state of motion, and consequently possesses energy, but under the conditions of hypothesis the quantity is less than any assignable finite magnitude, that is to say, *infinitesimal*, but the motion remaining in the fluid and the continued energy expenditure are of zero value *considered as infinitesimals of the same order*. Therefore, adopting a method of expression common in mathematical work (but not so frequently employed in direct physical demonstration), we may say that if we take as hypothesis a *small finite load*, so that the actual motions of the fluid be *small finite quantities*, the expenditure of energy in sustaining the load will be zero, *neglecting small quantities of the second order*.

§ 116. Interpretation of Theory of Aeroplane of Infinite Lateral Extent.—The system of flow deduced in the foregoing article in the case of an aeroplane of infinite lateral extent in an inviscid and incompressible fluid is one that may be classified as a *conservative system*, the energy of the fluid motion being carried

[1] Relatively to the earth.

along and conserved just as is the case in *wave motion*. The motion round about the plane may thus be considered as a supporting wave. When the amplitude of motion becomes sensible, there is no doubt that the streamlines react on one another in a manner not accounted for by the form of field contemplated, *i.e.*, that of the quiescent state. Under these conditions the method of investigation is not strictly applicable, but there would appear no reason to doubt the validity of the main inference. This conclusion is confirmed by a subsequent investigation conducted on different lines.

Considered in the light of wave motion, the peripteroid system must be regarded as a *forced wave*, the aerofoil supplying a force acting from without.

§ 117. Departure from Hypothesis.—Before proceeding to the further investigation, it is of interest to note briefly the consequences of a departure from the initial hypothesis.

If we suppose the aerofoil to be of finite lateral extent, it is immediately obvious that neither the lines of force nor the lines of flow can be represented by a single section through the field. The former, being no longer constrained to lie in parallel planes, diverge laterally, some portion of them escaping, as it were, and passing round the ends of the aerofoil through the regions marked *o*, *o*, *o* (Fig. 67), in which *R* and *L* are the right and left-hand extremities of an aerofoil whose direction of motion is perpendicular to the paper. The fluid traversing the regions *o*, *o*, *o*, will have upward momentum communicated to it during the whole time that it is in those regions, and will be finally left in a state of upward motion.

Now, owing to this lateral spread of the ascending field forward of the aerofoil, the upward velocity imparted to particles in that region is less than the downward velocity imparted in the corresponding portion of the descending field, and the fluid crossing the medial line *P Q* (Fig. 64) will have, on the whole, a downward velocity. Similarly the downward momentum imparted by the descending portion of the field aft of $P\ Q$ will be greater than the upward momentum imparted by the corresponding ascending field aft of the aerofoil. Consequently the portion of the fluid traversing the regions f, f, f, f (Fig. 67) will be ultimately left with some residual downward momentum, which must be equal to the total upward momentum received by the fluid traversing the regions o, o, o, for otherwise there would be a

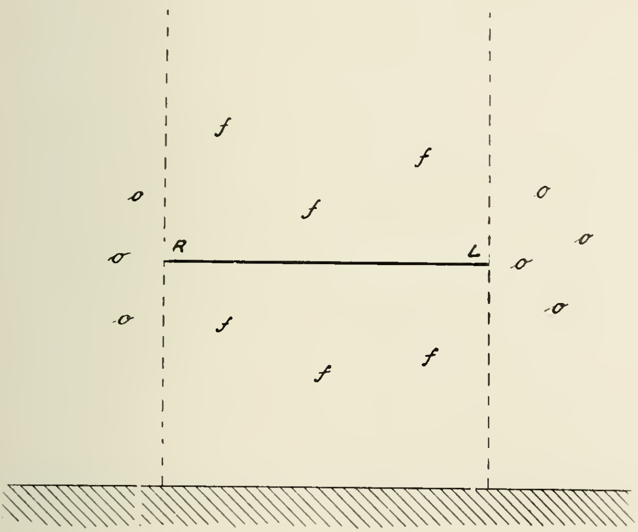

Fig. 67.

continual accumulation, or else attenuation, of the fluid in the lower strata of the atmosphere, which is impossible. (This otherwise constitutes an application of the principle of no momentum of § 5.) Thus in the case of a loaded aerofoil of finite lateral extent, there is a continual loss of energy occurring, and *a source of power is consequently necessary to maintain the aerofoil in horizontal flight.*

In addition to the residual vertical motions of the fluid, of which the causes have just been discussed, there must also be horizontal counter-currents formed simultaneously with those in a vertical direction, the horizontal and vertical motions being the horizontal and vertical components of the actual resultant motion of the fluid. We may regard the latter as in the main consisting of two parallel cylindrical vortices, having right and left-handed rotation respectively, which are being continually formed at the flank extremities (as in Fig. 61, reading this figure as an end-on presentation), whose energy is being continually dissipated in the wake of the advancing aerofoil.

From another point of view, this loss of energy may be looked upon as a gradual spreading out and dissipation of the wave (§ 116) on the crest of which the aerofoil rides, and it becomes necessary that the aerofoil should constantly renew the diminished wave energy in order to maintain sufficient amplitude and support the given load.

The first of these conceptions, *i.e.*, that of the vortex cylinders, is not, for a perfect fluid, compatible with *hydrodynamic theory*, for such vortex motion would involve rotation, and could not be generated in a perfect fluid without involving a violation of Lagrange's theorem (§ 71). In an actual fluid this objection has but little weight, owing to the influence of viscosity, and it is worthy of note that the somewhat inexact method of reasoning adopted in the foregoing demonstration seems to be peculiarly adapted, qualitatively speaking, for exploring the behaviour of real fluids, though rarely capable of giving quantitative results. The problem in three dimensions will be again examined after reviewing the subject on more rigid lines.

§ 118. On the Sectional Form of the Aerofoil.—We are at the present juncture in a position to draw certain elementary inferences as to the *form* of aerofoil appropriate to the motion of the air in its vicinity. The two aspects of form which are of most interest are, firstly, cross-section by a vertical plane in the direction of motion; and secondly, plan-form or projection on a horizontal plane.

The immediate function performed by the *sectional form* of the aerofoil is to receive a current of air in upward motion and impart to it a downward velocity, the whole air being dealt with possessing relatively to the aerofoil a superposed motion of translation. It would appear that any appropriate smoothly curved form, whose leading and trailing angles (Fig. 68) are conformable to the lines of flow, might be regarded as fulfilling the necessary conditions, the essential feature evidently being that neither edge shall give rise to a surface of discontinuity.

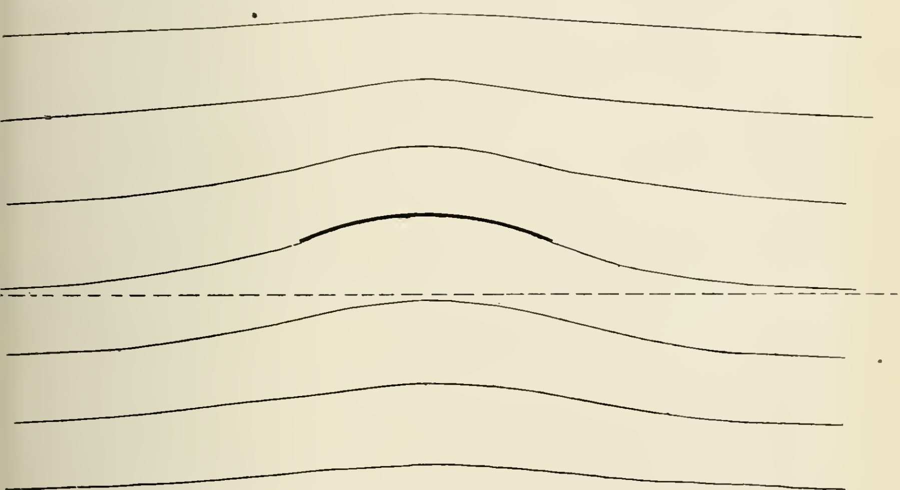

Fig. 68.

Since the amplitude of the motion may be regarded, for a fluid of given density, as a function of the load on the aerofoil and its velocity of travel, the steepness of the lines of flow must also be a function of these variables, and for a given sectional form of aerofoil there is some critical velocity at which the advancing edge may be taken as conformable. When the aerofoil is supposed of infinite lateral extent, then if the sectional form be made symmetrical, at the velocity at which the leading edge becomes conformable, the trailing edge will also be conformable. If, however, the aerofoil be of finite lateral extent, we do not know what the relation ought to be between the angles α and β, and we have as yet no means of ascertaining same, either for any particular point in the length of the aerofoil or generally for all points. The partial solution of this problem is reserved for a later chapter. It is probable that in nature the conformability of the trailing edge is substantially ensured by the extreme flexibility of feathered construction, an incidental advantage of this method being undoubtedly an automatic adaptability to variation of velocity and load.

§ 119. On the Plan-form of the Aerofoil: Aspect Ratio.—In the experiments of Professor Langley and others, planes of long, narrow plan-form, in pterygoid aspect, and at moderate angles, have always been found to give a greater lifting effort, *ceteris paribus*, than other forms, or than the same form moving end on. The reason of this is at once evident when it is considered that the amount of the fluid traversing the regions *o o o o*, Fig. 67, or "*stray field*," is relatively much less when planes of great lateral extent are employed, and every increase in the lateral extension of the plane makes the relative loss of field still smaller, the behaviour of the plane approaching more and more nearly to the ideal case in which the conservation is complete, and the plane reaps the benefit of the whole up-current generated.

Wherever flight has been successfully achieved, advantage has been taken of the influence of *aspect*; the *aspect ratio* varies amongst birds from about 4 : 1 (as in the *lark*, also *scops owl*) to about 14 or 15 : 1 (in the albatros). The wing spread with which Lilienthal successfully experimented had an aspect ratio of about 8 : 1, similar proportions being adopted in gliding machines subsequently by Pilcher, Chanute, and others. The author, experimenting in 1894, successfully employed a ratio of 13 : 1, and Phillips in his captive flying machine, about 1893, succeeded, by his "venetian blind" method of construction, in employing a ratio of more extreme proportion still.

§ 120. On Plan-form (continued): Form of Extremities.—The form of the extremities of an aerofoil exerts a considerable influence upon the dissipation of energy, irrespectively of the aspect ratio. It is evident that if, as a provisional assumption, we suppose the pressure distribution to be uniform over the whole area, the circulation will be much more rapid in the immediate vicinity of the edge than at some distance away; and since the fluid in circulation in the stray field represents energy lost, we can minimise this to a considerable extent by adopting a pointed, or acutely rounded, extremity, as in Fig. 69; so that the stray field is not contiguous to the edge of the aerofoil except in one spot at each extremity. If we neglect other factors that have weight in practice in determining wing form, and endeavour to rationalise on purely an aerodynamic basis, we can lay it down that for uniform load distribution, if we take the

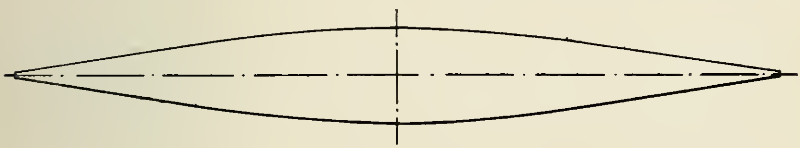

Fig. 69.

extreme wing tip as *origin*, the form of the wing extremity will be a surface that *can be generated by a straight line passing through the origin*. This law may be taken as holding good for such a length of the aerofoil at each end as may be regarded as inconsiderable in comparison to the total length, and follows from the absence of any *scale factor* in the problem; a surface as above defined may be regarded as a segment of the surface of an irregular cone.[1] It is possible that in a viscous fluid some departure from the form above prescribed may be anticipated from the fact demonstrated in the previous chapter, that the existence of viscosity is sufficient to give a *scale* to a fluid.[2]

In practice, it will be shown later in the work, the question of wing-form, especially with regard to the extremities, is not

[1] Compare §§ 190-192.
[2] Compare §§ 36 and 56.

decided by aerodynamic considerations alone, and that the question of equilibrium is involved.

It is evident that we are not bound to our assumption of uniform load distribution, and that if we suppose the pressure difference (between the under and upper surfaces) to be less towards the extremities, the latter may be made proportionately fuller without seriously disturbing the relative distribution of the stray field; we might thus take an elliptical form as a standard, with a pressure distribution appropriately proportioned. In general, the wing-plan of a bird has ordinates that approximate more or less closely to those of an ellipse. The discussion of the practical aspect of this question will be resumed in a subsequent chapter.

§ 121. Hydrodynamic Interpretation and Development.—We may recognise in the foregoing investigation (§§ 115 and 116) an elaboration of the theory initially put forward in § 90 (Chap. III), where the forces acting on the fluid were dealt with *in bulk*, instead of as in the present instance being studied in detail.

In § 90 it was shown that the disturbance peculiar to the neighbourhood of the aerofoil possesses angular momentum, and it was inferred that this being the case, the disturbance comprises a cyclic motion, for otherwise it must involve rotation, which is excluded by the nature of the hypothesis. We are consequently confined, in an *inviscid atmosphere*, strictly to the case where the aerofoil is of infinite lateral extent, for a cyclic motion is only possible in a multiply connected region.

The problem, then, from the hydrodynamic standpoint, resolves itself into the study of cyclic motion superposed on a translation. We have already devoted some attention to such a combination, and we have traced the field in a simple case for values of the functions ψ and ϕ, Fig. 48. In Fig. 70 we have the stream lines for this particular case plotted over a considerably greater area, the internal system of flow being replaced by a solid of substitution. We may look upon this figure as representing in section a theoretical wing-form, or *aerofoil*, appropriate to an inviscid fluid with its accompanying lines of flow; as such it is merely one of an infinite number of possible forms, its only virtue being that of representing the simplest possible case of *peripteroid motion*.

§ 122. **Peripteroid Motion.**—An infinite cylinder, of any sectional form whatever, divides infinite space into a doubly connected region, and in such a region cyclic motion becomes possible. From the hydrodynamic standpoint irregularity of contour is no detriment, as obstructing neither the cyclic motion nor that of translation. The consequence is that peripteroid motion is theoretically possible in the case of a cylinder of infinite extent, no matter what its cross-section. This conclusion applies naturally only in the case of the inviscid fluid; in a real fluid we are threatened with discontinuity. The position is analogous in every way to that of simple translation. In the inviscid fluid all bodies are of stream-line form, in real fluids only those that in their motion do not set up a discontinuity. Again, just as in the simple translation only certain simple cases are capable of solution by known analytical methods, so in peripteroid motion the cases capable of solution are very limited in number.

In order that a case of peripteroid motion should be solvable, the boundary conditions (both internal and external) must, generally speaking,[1] be such that their lines of flow for both translation and cyclic motion are separately known. The author has succeeded in plotting the stream lines in the following cases:—

Fig. 70, a filament of infinite lateral extent in an infinite expanse of fluid.

[1] A case, such as Fig. 70, is an exception. Here neither system is known separately for a cylinder the form of the shaded section. In a case of this description, where a body is substituted for a self-contained system of flow, we have an exception to the fact stated.

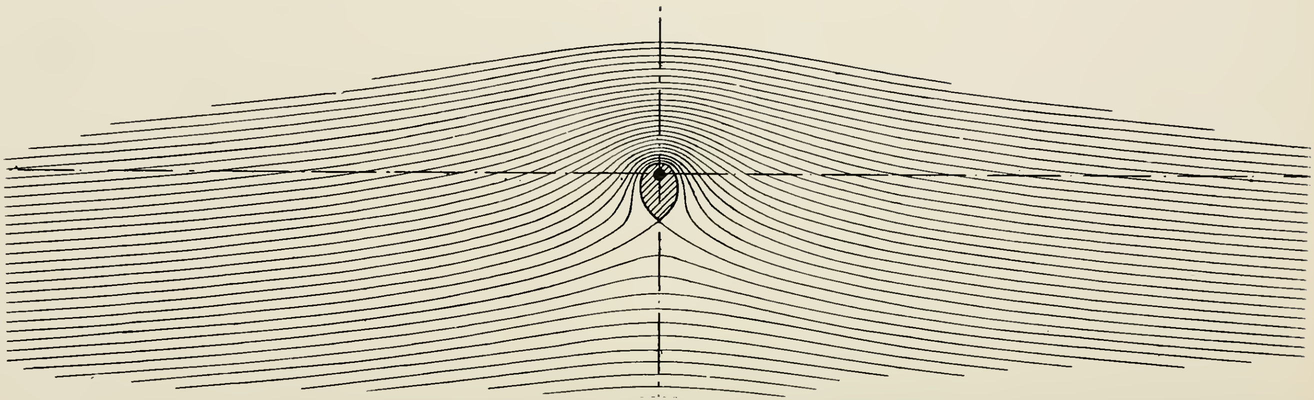

Fig. 70.

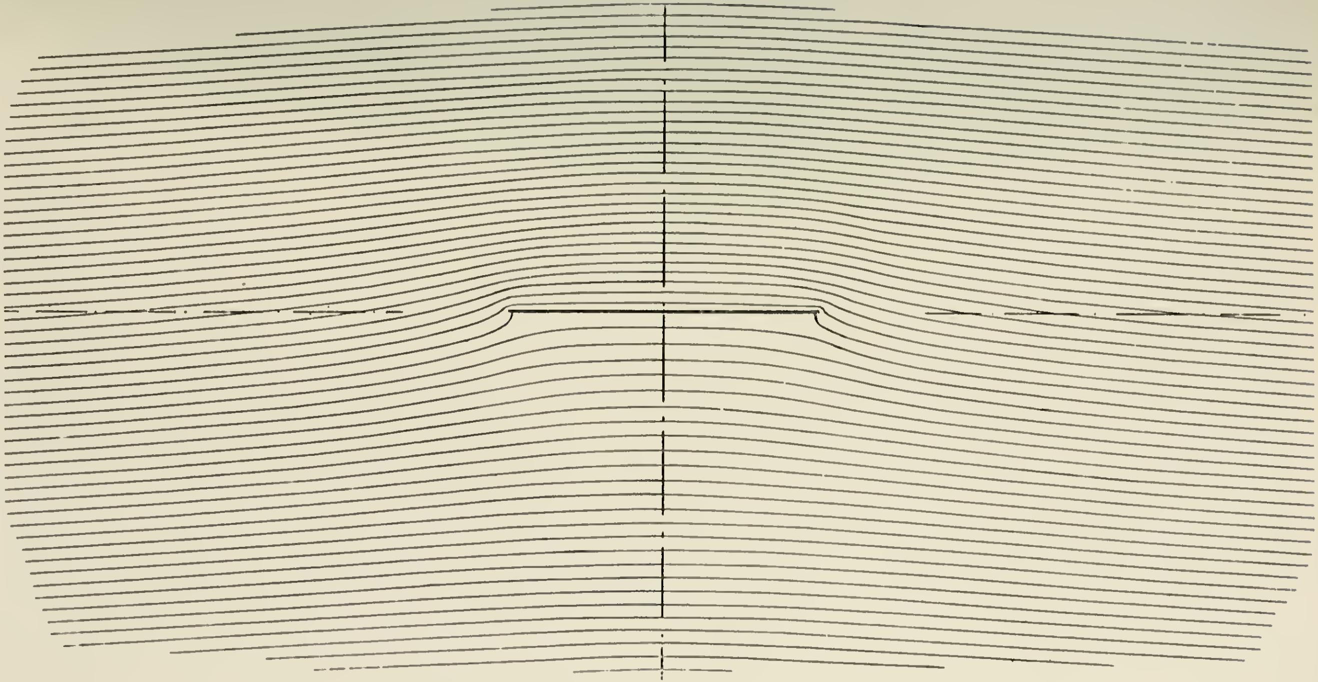

Fig. 71.

Fig. 71, a plane of infinite lateral extent moving edgewise, may be taken as an aeroplane at evanescent angle. Fluid infinite.

Fig. 72, the same as Fig. 71, but with more powerful cyclic component, showing form of motion in greater detail.

Fig. 73, combined system due to two superposed planes, separated $1\frac{1}{8}$ times their width. Planes and fluid infinite.

Fig. 74, elliptical cylinder of infinite lateral extent, in infinite expanse of fluid.

Fig. 75, an aeroplane of evanescent angle in vicinity of boundary surface.

§ 123. Energy in the Periptery.—A body in motion in a fluid is known to carry with it kinetic energy due to the fluid disturbance in addition to that due to its proper mass (§§ 81, 84). A superposed cyclic motion adds to the energy so carried.

A cyclic motion around a cylinder or cylindrical filament, or round about a plane, in an infinite expanse of fluid contains an infinite quantity of energy (§ 85), and the resulting peripteroid motion for these cases will consequently require an infinite quantity of energy for its production. We must consequently regard Figs. 70, 71, 72, 73, 74 in the light of types of motion, rather than an actual form of motion that we could produce if the circumstances of hypothesis were materialised. If, however, we limit the expanse of fluid by a boundary, such as in Fig. 75, the energy of the cyclic motion immediately becomes finite, for the number of squares is limited (§ 86), so that the flow as here depicted is not open to the same objection.

The quantity of energy in the particular case given in Fig. 75 is equal to that of a body of fluid moving with the aeroplane, whose area is approximately one-seventh of that of the square on the aeroplane section.

The quantity of energy contained in peripteroid motion, and its relation to the load supported, is a matter that awaits more complete investigation.

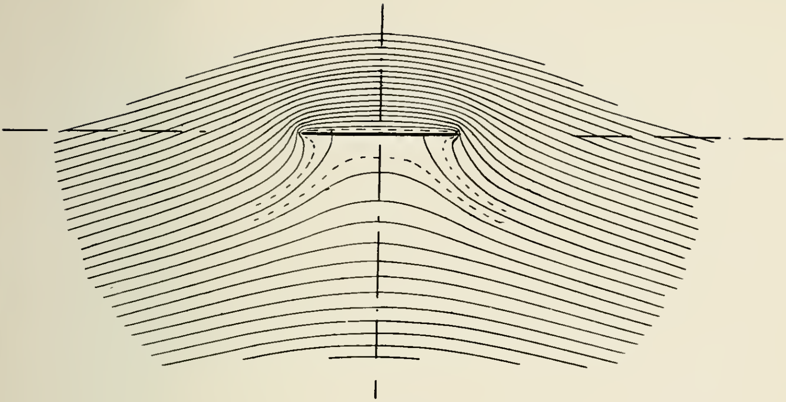

FIG. 72.

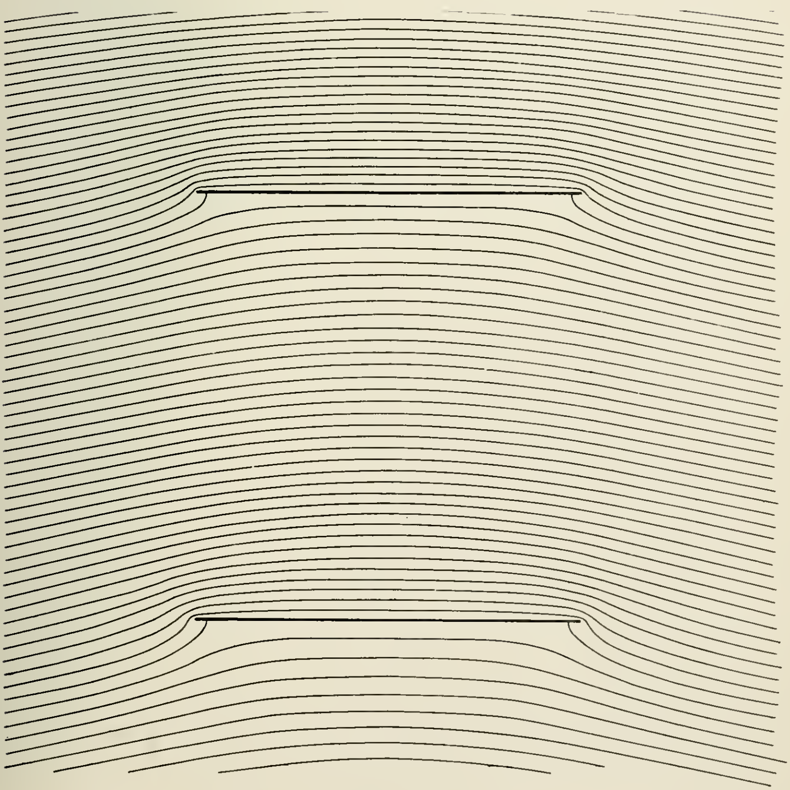

FIG. 73.

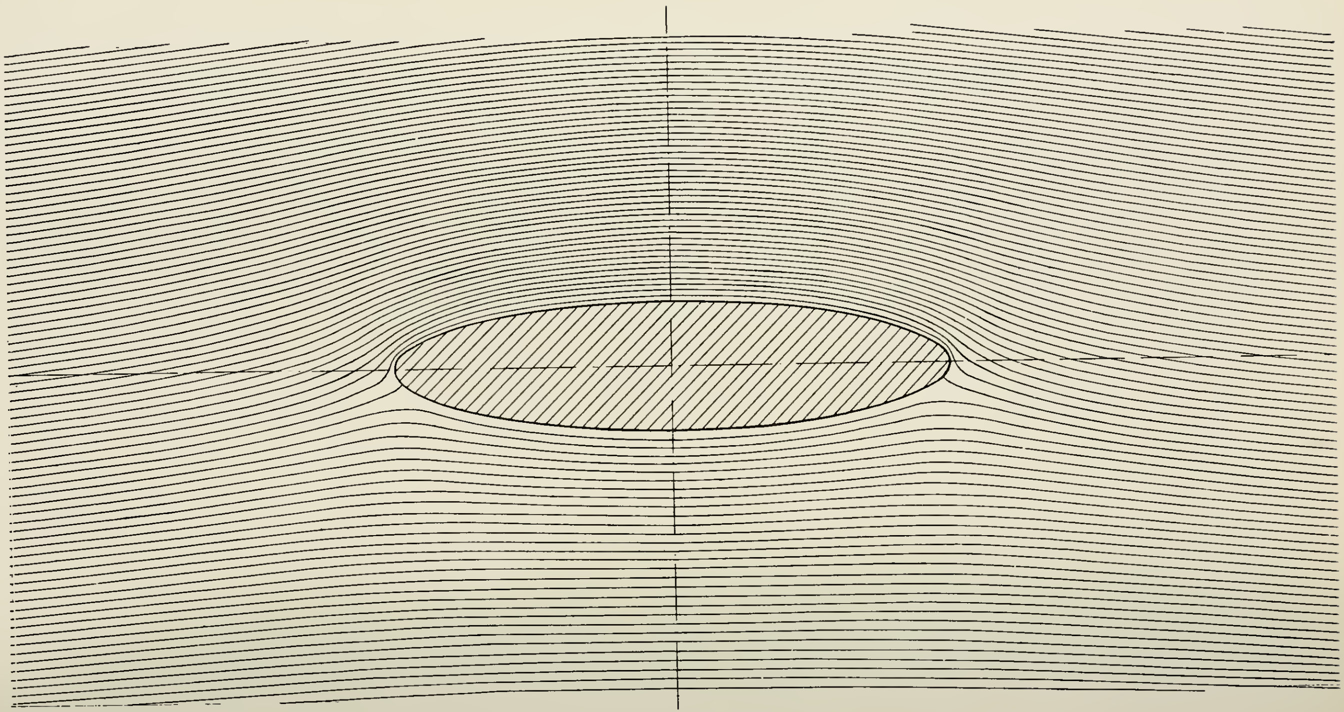

FIG. 74.

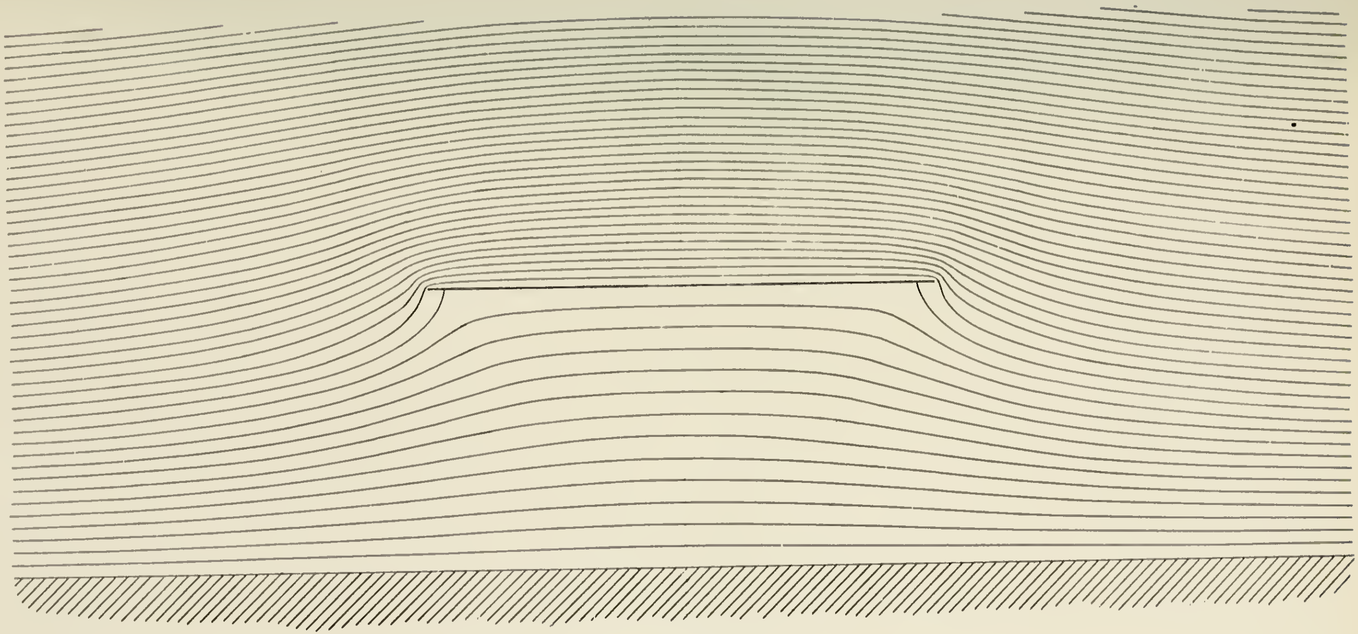

Fig. 75.

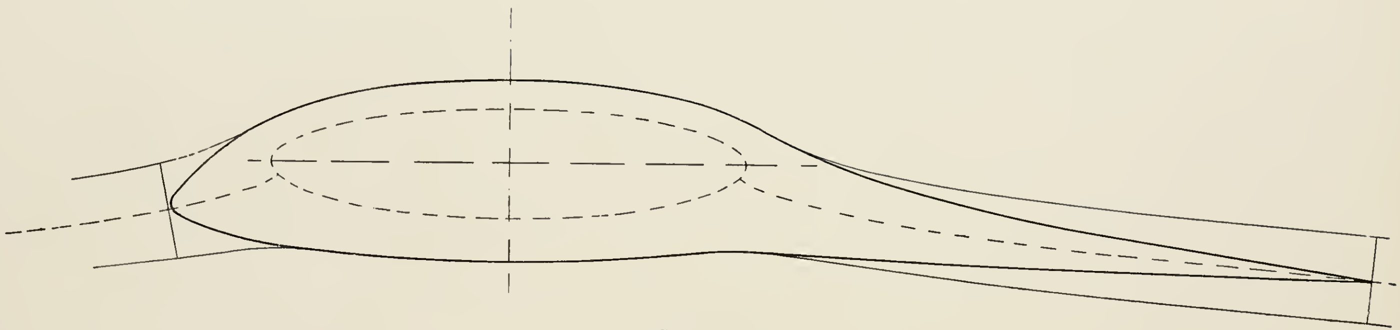

FIG. 76.

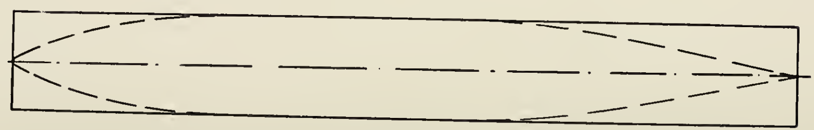

FIG. 77.

§ 124. Modified Systems.—In the examples given in Figs. 70, 71, 72, 73, 74, there are in all cases abrupt motions of the fluid at certain points, such as could not occur in practice where a real viscous fluid such as air is concerned; the stream lines that most nearly fulfil the necessary conditions are those belonging to the elliptical cylinder (Fig. 74).

If we select from Fig. 74 a pair of stream lines possessing the requisite smoothness of curvature as the boundary of a supposed aerofoil, and, having truncated the fore and aft extremities,

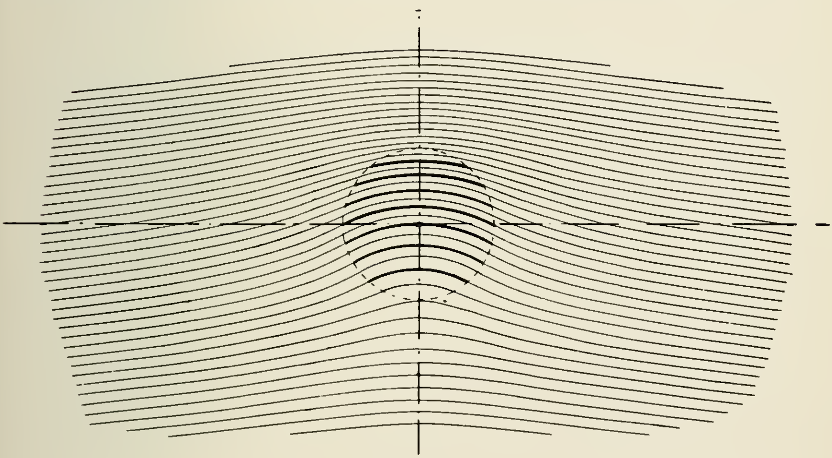

FIG. 78.

proceed to whittle away the abruptness of the ends so formed (Fig. 76), we obtain a possible wing section whose form, derived entirely from theoretical considerations, bears an unmistakable resemblance to an actual section taken through the thick of the wing of one of the larger soaring birds. The whittling process is supposed carried out just as would be done in the case of a plank, originally sawn with square edges, to which it is desired to give a stream line form (Fig. 77).

A different and perhaps not quite so legitimate subterfuge is employed in Fig. 78, in which the space enclosed within the dotted line is supposed to contain uniform *rotation*. This requires that the load should be distributed throughout the region in question (compare § 92), a condition that could be only *approximated* in practice by the employment of a number of surfaces, such as indicated diagrammatically by the stouter lines shown in the figure. The form of these surfaces for the conditions stated is that of a series of concentric cylindrical sections.

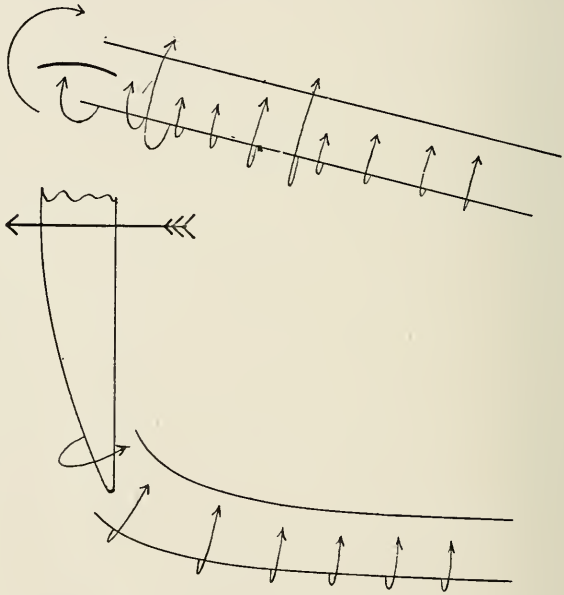

FIG. 79.

§ 125. Peripteroid Motion in a Simply Connected Region.—The problem presented in the case of an aerofoil of finite lateral extent is, from the present standpoint, one of some difficulty, inasmuch as the region under these circumstances becomes *simply connected*, so that cyclic motion can no longer exist, and rotation in some form constitutes the only solution. It is, of course, conceivable that flight in an inviscid fluid is theoretically impossible.

Let us first study the case of a *viscous fluid*, and then, by supposing the viscosity to become less and less, endeavour to approach the conditions of the *inviscid*.

We have seen in § 117 that the lateral terminations of the aerofoil give rise to vortex cylinders, which trailing behind gradually dissipate their energy in the wake. Such a supposition presents no difficulty in viscous fluid, for the core of the vortex cylinders can then be formed of a mass of fluid in rotation.

Now we know that two parallel vortices, such as we have here, possessed of opposite rotation, in the first instance attract one another, and by their mutual interaction move through the fluid parallel to one another in the direction of motion of the fluid that lies between them (§ 93). Consequently in the present instance they will *precess* downwards as fast as they are formed, so that the aerofoil and its accompanying vortex train will appear somewhat as shown diagrammatically in elevation and plan in Fig. 79.

But if the dissipation of the vortex motion takes place sufficiently slowly, as when the viscosity of the fluid is not great, the vortices may persist until they reach the level of the ground. Under these circumstances one of two things will happen: either the vortices will spread apart as they approach the ground surface, each acting under the influence of its own "reflection" in the well known manner, or the ends of the vortices will attach themselves to the surface in the manner suggested by § 93.

If it be supposed that the aerofoil and its load were created in some upper region, and set in motion away from the earth's surface, the former assumption would be perhaps the most academically correct: if, however, we suppose the loaded aerofoil to be launched from the earth beneath, the vortices would naturally grow out from the surface, and would remain attached to the surface as they travel with the aerofoil to which they belong.

In the case of real fluids, the existence of these vortices can be traced experimentally by the employment of an aerofoil *under* *water and inverted* (Fig. 80), the pressure region being on its upper surface, and the vortices being evidenced by the dimples in the surface of the water. This experiment shows that in practice the vortices are continually breaking up and being left behind as fragmentary eddies. If the experiment is tried in a comparatively narrow vessel the eddies are actually found to have retrograde motion, owing to the influence of their own "*reflexion*" in the sides of the vessel. If the experiment were tried in an open expanse of water on a large scale it would probably give more perfect results.

It would appear probable that in a fluid of very small viscosity vortices springing from the extremities of the aerofoil and terminating on the boundary surface may be permanent; in fact, we might regard the whole system as a single-vortex filament, with both its extremities situated on the boundary, and enclosing the aerofoil as an incident. Following out this idea, we should obtain, for an inviscid atmosphere, a system consisting primarily of a vortex hoop or half-ring, loaded in the centre by the aerofoil (Fig. 81), and whose energy will be perfectly conserved, the aerofoil and its supporting vortex lying in a plane at right angles to the direction of flight. Such a system in a fluid that is truly inviscid would be uncreatable and indestructible, just as in such a fluid a vortex ring is uncreatable and indestructible. The system of static forces called into play is represented diagrammatically in Fig. 82, in which the tension due to the vortex motion is represented by an irregular polygon following the vortex core, the forces at right angles being those due, on the one hand to the load on the aerofoil, and on the other to the cyclic motion round the vortex core in translation,

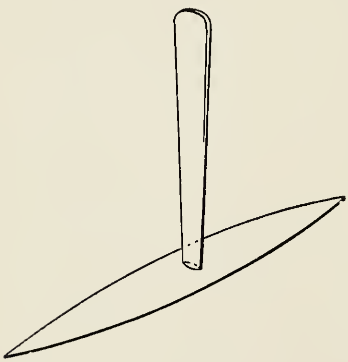

Fig. 80.

that is, the force that prevents an ordinary vortex ring from collapsing on to itself.

Pending the complete hydrodynamic investigation of such a system as above sketched out, it must be regarded somewhat in the light of a speculation in which there is nothing actually

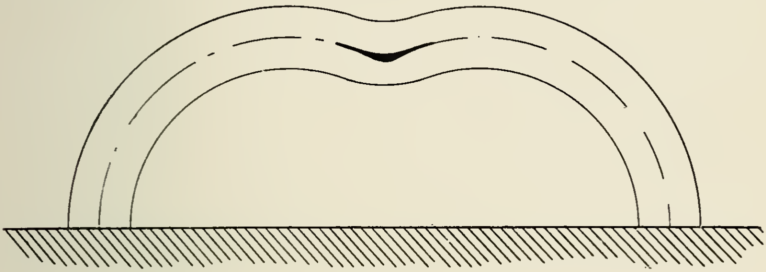

FIG. 81.

improbable. The conception suggests that if we had been called into existence surrounded by an atmosphere destitute of viscosity our natural method of locomotion would have been to glide horizontally sustained on the crest of a vortex hoop, a structure

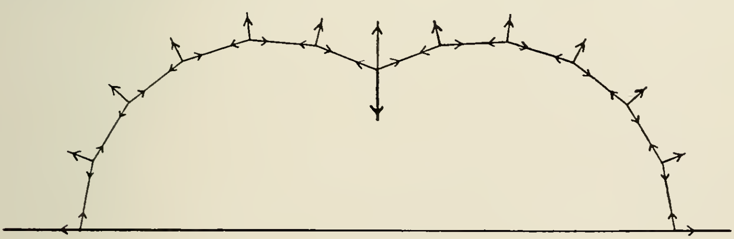

FIG. 82.

which from its immutability would require to be specially created at birth, and would after death continue to pervade the world for all time like a disembodied spirit.

§ 126. Peripteral Motion in a Real Fluid.—In dealing with a real fluid the problem becomes modified; we are no longer under the same rigid conditions as to the connectivity of the region.

The whole subject of cyclic motion in the case of a viscid fluid has not been thoroughly investigated. It is evident that to a certain extent the restrictions proper to the inviscid fluid must apply, but since we can generate rotation we are able to induce vortices with a freedom not possible when viscosity is absent.

Basing our argument on the facts as already ascertained, it is evident that if we continuously generate vortices at the right and left hand extremities of the aerofoil, as in Fig. 79, we can regard these vortices as forming in effect, taken in conjunction with the

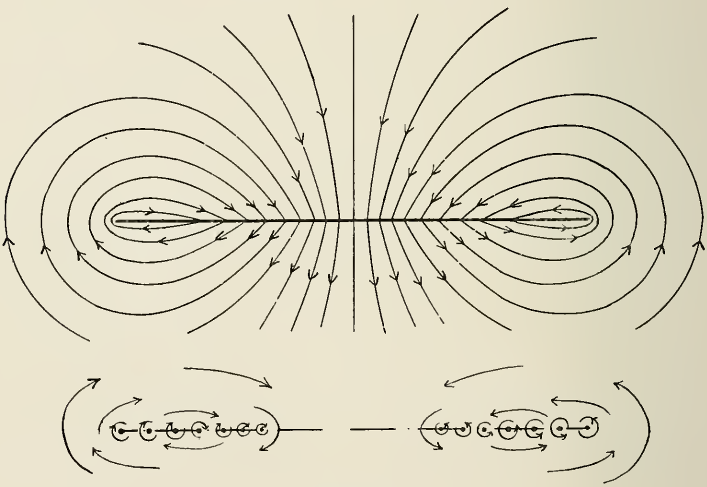

Fig. 83.

aerofoil itself, an obstacle to connectivity, so that, although the vortex dies away after a while, it persists as long as is necessary to permit of a cyclic system being established and maintained.

It is probable that these terminal vortices do not each actually consist of a single vortex but rather of a multiple system of smaller vortices; especially should this be the case with the larger birds, and similarly for mechanical models of any size.

We can conceive that these vortices are formed after the manner indicated in Fig. 83, in which an aerofoil is represented in end elevation with the flow indicated diagrammatically. We may suppose that the air skirting the upper surface of the aerofoil has a component motion imparted *towards* the axis of flight, and that skirting the under surface in the opposite direction, so that when the aerofoil has passed there exists a Helmholtz surface of gyration. This surface of gyration will, owing to viscosity, break up into a number of vortex filaments or vortices after the manner shown.

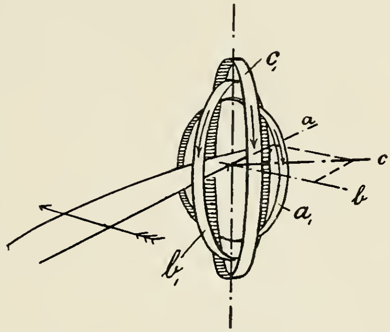

Fig. 84.

§ 127. Peripteral Motion in a Real Fluid (continued).—The cyclic flow of the vortices to the right and left hand of the aerofoil finds itself superposed on the main cyclic system of the aerofoil, so that the axes of these vortices will not be parallel to the axis of flight as might be supposed, but will take up a resultant direction and may be conceived to spread out as shown in Fig. 85. The compounding of two cyclic systems into a resultant system is illustrated diagrammatically in Fig. 84, in which the circle *a a a* represents the main cyclic system, that whose supporting reaction is concerned in sustaining the load; *b b* represents the cyclic system of one of the vortex filaments, and *c c* the resultant.

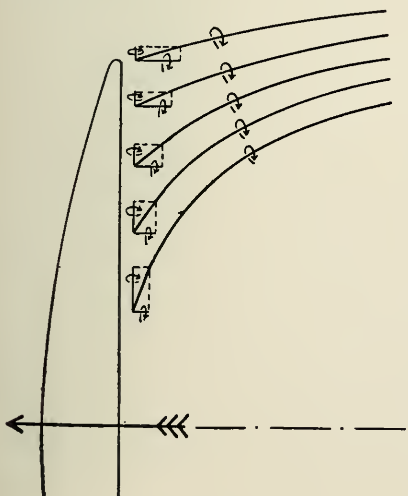

Fig. 85.

Representing diagrammatically the relative strengths of the cyclic systems as the sides of a parallelogram (Fig. 85), we arrive at an indication of the manner in which the vortices will spread as they are left behind by an aerodrome in flight.

Following the matter further we may represent the interaction of the vortices on each other in the manner shown in plan in Fig. 86. This figure is merely a diagram, the motion indicated

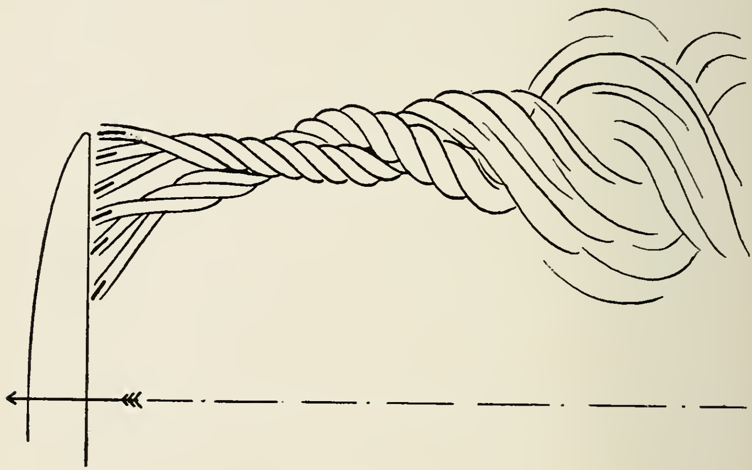

Fig. 86.

being based on the known properties of vortices (§ 93). The filaments will evidently wind round one another like the strands of a rope, being involved in common in the resultant cyclic disturbance. The two vortex *trunks* springing respectively from the right and left hand wings, owing to their rotation being opposite, do not wind round each other but precess downwards as in Fig. 79. The motion is represented as becoming incoherent in Fig. 86, as undoubtedly must sooner or later be the case.

CHAPTER V.

THE AEROPLANE. THE NORMAL PLANE.

§ 128. **Introductory.**—Any material plane, that is to say any thin rigid plate bounded by parallel plane surfaces, when propelled through the air or held stationary in air in motion, experiences a reaction of greater or less magnitude. Any such "plane" is, from the manner of its employment, termed an *aeroplane.*

Theoretically an aeroplane is regarded as being material and rigid without possessing thickness. In practice, a certain amount of thickness being necessary, the edges may either be cut square, as in the planes employed by the late Professor Langley, in which case an allowance requires to be made for the edge effect, or, the edges may be carefully bevelled and rounded off, so that the aeroplane becomes an equivalent body of stream-line form, in which case it is believed that no allowance is required.

The study of the aeroplane may be said to form the elementary basis of experimental aerodynamics as relating to the problem of flight. Whilst laying due stress on this fact, it may be pointed out that the importance of aeroplane study consists in its educational value and its bearing on certain subsidiary problems, rather than in the direct application of the aeroplane to the main function of flight, *i.e.*, the support of the weight. This statement might appear somewhat unexpected, but it may be explained at the outset that the author does not employ the term *aeroplane* outside its correct signification, that is to say, to denote other than a true or *plane* aeroplane; the misuse of the word being avoided by the introduction of the term *aerofoil,*[1] to denote a

[1] From Gr. ἀέρος and φυλλον (lit. an *air-leaf*).

supporting member, or organ of sustentation of undefined form. Thus a *plane aerofoil* is an aeroplane, or a *pterygoid aerofoil* is an aerofoil of wing-like form.

There are cogent reasons why the aeroplane should take foremost place in the matter of experimental study. It is recognised as essential to the inductive mode of investigation that, *whenever possible*, one of the conditions of experiment, and one only, should be changed at a time, and it is primarily on this ground that the aeroplane recommends itself. The aeroplane is possessed of a geometrical definiteness that admits of no ambiguity; a specified contour form, making a definite angle with, and presenting a definite *aspect* to the line of flight, constitutes (for any given velocity) the whole of the factors by which the conditions of experiment are defined. Beyond this there is (skin-friction apart) a certain obvious relationship between the *pressure components about the co-ordinate axes*, and the *angle of flight*, that forms a valuable and instructive link in the interpretation of experimental results.

§ 129. Historical.—Our knowledge of the aeroplane to-day is the result of the work of a number of investigators. The exact date at which the study of the subject was seriously taken in hand is in doubt; a certain amount of experimental work on the resistance of bodies in the air is known to have been done early in the eighteenth century, notably by Sir Isaac Newton (1710), and Dr. Desaguliers (1719), whose observations are, however, believed to have been confined to the motion of spherical bodies. Newton also extended his researches to the theoretical study of bodies of different forms in a hypothetical *medium* (ref. § 2), and showed that the theoretical and experimental results are not altogether out of harmony in spite of the unreal nature of his hypothesis. Newton further attempted to solve the problem of the normal plane in an incompressible continuous medium (*Principia*, prop. xxxvii. and cors. 7 and 8, prop. xxxvi). These propositions, resting as they do on the supposition of the congealing of portions of the fluid, are known to be unsound, but the results are not without interest.

The next experimental records chronologically are those of Robins (the inventor of the experimental device known as the *whirling table*), about the middle of the eighteenth century, and Charles Hutton in and about the years 1787—8; whilst among the most recent may be mentioned the systematic researches of S. P. Langley, 1888—90, and the investigations of W. H. Dines of about the same period. An abridged account of the most important of these investigations, with some criticism of the methods and conclusions, is given in a subsequent chapter[1] devoted to experimental aerodynamics.

§ 130. The Normal Plane.—Law of Pressure.—The simplest case of the aeroplane is that in which the direction of motion through the air is at right angles to its surfaces.

Even under these simple conditions the determination of the pressure-velocity law has not been made without some difficulty, and although the approximate form of the expression, P varies as V^2, was correctly given by Hutton, Smeaton and others more than a century ago, it is only of recent years that the *constant* connecting the two sides of the equation has been ascertained with any degree of certainty, and that with a possible error of five per cent. or so. Writing the expression in the form— $P = k\,V^2$, the value of k is variously given by different authorities as from ·00166 to ·0023 where P is in *pounds per square foot*, and V is in *feet per second.*

The experimental basis of the law of the Normal Plane is two-fold; tests of wind pressure at known mean velocity, and experiments on the resistance to motion of planes through still air. At first sight there might appear to be no fundamental distinction between these two methods; the difference might be thought to be merely one of relative motion; owing, however, to certain considerations that require to be taken into account, the

[1] Chap. X.

results obtained by the two methods are strikingly different, and the discrepancy in the value of the constant as given by different writers may be to a certain extent explained.

§ 131. Wind Pressure Determinations.—One of the characteristics of the aerial disturbance which we know as *wind* is the continual fluctuation both as to direction and velocity; this characteristic is so well known as to have found expression in the vocabulary of every civilised nation—"gust of wind," "coup de vent," etc. Wind may be said to consist of a general motion of translation with a superposed motion of turbulence (§ 37), the result being that at no point does the velocity or direction remain constant for any length of time.

One immediate consequence of this variability is that for a wind of known mean velocity = V, the mean value of V^2 is higher than would be the case if the problem were one of uniform air current having the same mean velocity, and therefore the pressure (which depends upon V^2) will also be higher. If we neglect the secondary effect due to the components of motion of the air in directions parallel to the pressure plane (§ 146 *et seq.*), so that the mean pressure on the plane is due only to the normal component of motion of the wind, then it would appear that the pressure will be proportional to the energy per unit volume; for dimensionally:—

$$\text{Pressure} = \text{Force}\ /L^2$$
$$\text{Force} = \text{Energy}\ /L$$
$$\therefore \quad \text{Pressure} = \text{Energy}\ /L^3.$$

That is to say, the *pressure is proportional to the energy per unit volume.*

Now the average energy per unit volume in the wind is the sum of the separate energies of mean velocity and of turbulence (the latter for our present purpose being reckoned only in respect of motion in the direction at right angles to the pressure plane), and in a wind possessing such energy of turbulence, the mean pressure will be greater than would be the case for the simple air current in the proportion that the sum of the energies bears to the energy of mean velocity.

The validity of the above reasoning is unquestionable if we are dealing with a fluid whose properties are those of the Newtonian medium; the energy of turbulence being represented by a variation in the individual velocities of the particles such as will not affect their mean velocity. It is, however, open to question whether it applies rigidly in the case of a fluid possessed of continuity.

Another standpoint from which we may view the present problem is that the turbulence, by effecting a rapid transference of momentum from one part of the fluid to another, acts *in effect* to augment the apparent viscosity, and in this way adds to the pressure reaction. In any case experiments made on a fixed plane or other body in moving air, cannot be regarded as valid when the conditions are reversed.

Beyond the above there are certain considerations of a practical nature that tend to further invalidate wind pressure measurements as representing aeroplane resistance. It is probable that the maximum pressure on a plane under given wind conditions is not in proportion to the area exposed, and that a small plane is liable to greater extremes, and where a maximum record is made, the *absolute area* exposed becomes an important factor in determining the *pressure per unit.*

§ 132. Still Air Determinations.—Under the conditions of experiment in still air, none of the foregoing considerations apply, and it may be safely asserted that the resistance per unit area is approximately proportional to the square of the velocity and is almost independent of the size of the plane. There is some doubt as to the exactitude of the V^2 law, as in all similar cases of fluid resistance, and it is likely that this doubt will remain until the methods of experiment have undergone refinement; on the one hand, if there is a departure, existing method is too crude to determine its nature, on the other hand it has been shown by Allen (§ 35 *et seq.*), that where the V^2 law rigidly applies the resistance is entirely independent of viscosity, a result that would appear to be highly improbable.

It has been proved from the behaviour of projectiles in flight that the V^2 law breaks down when the velocity of sound is approached, and without doubt this applies also to an aeroplane when a similar velocity is reached. The defect that manifests itself at these high velocities is that the pressure becomes considerably greater than the law would indicate, or as it may be expressed, the value of the index increases, the expression being written: $P = k\,V^n$. On the other hand, at very low velocities at which the influence of viscosity makes itself felt the law becomes modified in the opposite direction, the value of the index diminishes.

It is probable that in actuality these two influences correct one another over a fairly wide range, so that the V^2 law may become a far closer approximation than would otherwise be the case.

§ 133. Quantitative Data of the Normal Plane.—The following are the generally accepted data of the Normal Plane, the authority being stated where known :—

Wind Pressure.—$P = k\,V^2$ where the constant $k = \cdot 0023$, P is in pounds per square foot, and V feet per second.

The value of k given is that usually accepted, and will be found in the majority of text-books, also in the "Encyclopædia Britannica" under the article on "Wind." Molesworth, in his "Pocket Book," gives the figure ·002288, but his authority is not disclosed, neither are particulars given of the method by which accuracy has been obtained to so many places of decimals.

Still Air Data.—Form of expression and units as before. Hutton is quoted as giving $k = \cdot 0017$. This result at the time of his experiments (1787—8) must be considered quite remarkable, in view of the fact that one of the most recent determinations, the corrected mean of a great number of experiments made by Professor Langley, exactly a century later, gave an almost identical result.

Langley, in presenting his final result, $k = \cdot 00166$ as the corrected mean of his experimental records, states that the possible errors of experiment are such as to leave a probable uncertainty of about 10 per cent. The temperature and pressure corresponding to the above value are given as 10 degrees C. and 736 m.m. mercury; if we reduce to sea level we obtain Hutton's result, $k = \cdot 0017$, almost exactly.

Dines[1] has shown that the pressure depends not only upon the velocity but also upon the *shape* or "contour form" of the plane, and that the pressure is least for planes of compact outline, such as a square or circular disc. In his experiments he obtained values for a rectangle 16 inches × 1 inch greater in the proportion of 8 to 7 than for a square of equal area. The value of k given by Dines for planes of compact form is about 6 per cent. below that of Langley; the latter value is approximately equal to Dines' result for a rectangle of 4 : 1 ratio. This 6 per cent. difference is an actual disagreement. The planes employed by Langley for his determination were of square form.

§ 134. Resistance a Function of Density. — Employment of Absolute and Other Units. — In order that the expression $P = k V^2$ should be *dimensional* the constant k must include a quantity of the dimensions $\frac{m}{l^3}$. This can be eliminated by introducing the density of the fluid into the expression.

Employing *British Absolute Units*, let :—

P = pressure in *poundals per square foot.*

V = velocity feet per second.

ρ = density of fluid, lbs. (mass) per cubic foot.

C = constant.

[1] Quarterly Journal, Royal Met. Soc., Vol. XV., No. 72, October, 1889.

The expression then becomes :—$P = C \rho V^2$, in which $C = 32{\cdot}2\ k/\rho$, and for air[1] at 10 degrees C. and 760 mm. pressure we have $\rho = {\cdot}078$, whence,—

$$C = \frac{32{\cdot}2 \times {\cdot}0017}{{\cdot}078} = {\cdot}7.$$

The equation thus becomes :—

$$P = {\cdot}7\ \rho\ V^2.$$

The equation is identically the same in C.G.S. *absolute* units, and the constant is of the same value ; that is to say, P = *dynes per square c.m.*, V = *c.m. per second* and ρ *grammes per cubic c.m.*

If we express P in *grammes per square c.m.*, and V in *metres per second*, and substitute for ρ for air at 10 degrees C., we obtain the equation in the form :—

$$P = {\cdot}009\ V^2.$$

If the velocity is given in English *miles per hour* it is sometimes convenient to have the expression in the form :—

$$P \text{ (pounds)} = \frac{V^2}{275}.$$

§ 135. **Fluids other than Air.**—If the whole physical properties of a fluid were represented by the symbols in the equation, or if, the equation being as it is, the fluid were incompressible and of zero viscosity, the constant C would be the same for different fluids.

The experimental determination in the case of sea-water has been made by Captain Beaufoy, and independently by R. E. Froude, the results being in close agreement. In absolute units we have :—$P = {\cdot}55\ \rho\ V^2$, that is to say, the value of the constant is $\frac{55}{70}$ or approximately four-fifths of that in the case of air.

This difference is undoubtedly due to the lower kinematic viscosity of water, which is less than air in the ratio of 1 : 14. The nature of the relationship connecting the function *kinematic viscosity* and the changes in the value of the constant, is not very

[1] From the determination of Regnault.

clear; the existence of such changes shows the form of the expression to be inexact, for, according to Allen (§§ 35 and 42), under these circumstances the V^2 law cannot strictly apply. It might, without departing from the form of the expression, be possible to establish an empirical relationship, and it is in any case of interest to endeavour to ascertain the probable magnitude of the constant for the particular case when viscosity becomes vanishingly small.

There is no fluid known of which it can be said that viscosity is a negligible quantity; neither is it possible to deduce from the data of known fluids what the behaviour of such a fluid would be. We have consequently to fall back on pure theory.

§ 136. Normal Plane Theory Summarised.—Several methods of computing the pressure on a normal plane have been proposed; up to the present none of these can be considered entirely satisfactory.

1. *The Method of the Newtonian Medium.*—The theory of the Newtonian medium has been already discussed (§ 4); it has been shown that on this hypothesis we have two possible results: (*a*) if the particles are elastic, $P = 2 \rho V^2$; (*b*) if the particles are inelastic, $P = \rho V^2$.

Both these results are higher than that given by experiment for a viscous fluid, a defect that is due to the faulty hypothesis, the Newtonian medium possessing no continuity. Newton was fully conscious of this fact.

2. *The Newtonian Method* (Book II., Section VII. prop. xxxvii.).—In this proposition,[1] Newton arrives at a result for a fluid possessing continuity the equivalent of which is:—

$$P = \cdot 25 \rho V^2$$

3. *The Torricellian Method* is here so named merely as a matter of convenience as being based on the Torricellian principle, and not as due to Torricelli himself.

In a continuous fluid, the theorem of Torricelli, which is

[1] See § 129.

founded on the *Principle of Work*, shows that a given pressure is capable of generating a certain definite velocity in the fluid; thus, representing the pressure of the fluid by its hydrostatic head, the latter gives the height through which a body would require to fall in order to acquire the corresponding velocity. If we arrogate that the converse is true, *i.e.*, that a given velocity is capable of generating the corresponding pressure, and that the conditions present in the case of the Normal Plane are such that this corresponding pressure will be generated, then we obtain the result: $P = \cdot 5 \, \rho \, V^2$, for, if s = "head," we have: $s = \frac{V^2}{2\,g}$ and mass whose weight constitutes pressure $= \rho \, s$, or

pressure $= g \, \rho \, s = g \, \rho \, \frac{V^2}{2\,g} = \cdot 5 \, \rho \, V^2$.

4. *The Helmholtz Kirchhoff Method.*—This method is based on the theory of Discontinuous Motion (Chap. III., § 97); the solution is only known in the case of a lamina bounded by parallel lines of infinite length; in this case the expression is: $P = 44 \, \rho \, V^2$.

The Helmholtz theory of discontinuous motion is in all probability the correct theory of the fluid *whose viscosity is vanishingly small;* and the above result may therefore be taken as rigidly accurate. It is unfortunate that the mathematical difficulties of this method have only been overcome in a few isolated cases.

To summarise, we have :—

(1) Newtonian medium (*a*) elastic particles	$C = 2$	All shapes.
Newtonian medium (*b*) inelastic particles	$C = 1$	,, ,,
(2) Newtonian method, prop. xxxvii.	$C = \cdot 25$	,, ,,
(3) Torricellian method . . .	$C = \cdot 5$	,, ,,
(4) Helmholtz-Kirchhoff . . .	$C = \cdot 440$	infinite lamina.

And experimental determinations in ordinary viscous fluids as follows :—

Air (Hutton)	$C = {\cdot}7$	Shape not stated.
,, (Langley) ...	$C = {\cdot}7$	Square planes.
,, (Dines)	$C = {\cdot}66$	Circular and other "compact" forms.
,, (Dines)	$C = {\cdot}76$	Rectangular lamina 16 : 1 ratio.
Water (Beaufoy, Froude)	$C = {\cdot}55$	Shape not stated.

§ 137. Deductions from Comparison of Theory and Experiment.— The method of the Newton medium may be dismissed on the grounds of faulty hypothesis; the prop. xxxvii. method may be discarded as being certainly unsound; the Torricellian method is based on a tacit assumption that the fluid in proximity to the front face of the plane is destitute of velocity, which we know is not true, except at one point or on one line. The Helmholtz method alone stands on a scientific basis, and at present this gives a result in but one special case.

We are in want of data; let us assume data and develop the method. The results can be corrected for more reliable figures when such have been ascertained.

Data assumed:—

Helmholtz' result for infinite lamina . . $C = {\cdot}44$.

Dines' determination for plane 16 inches × 1 inch assumed as for infinite lamina . $C = {\cdot}76$ (Air).

Beaufoy's result augmented 10 per cent. for infinite lamina. (Water) . . . $C = {\cdot}60$.

These values are plotted in Fig. 87, in which abscissæ represent viscosity (kinematic), and ordinates values of C; that is, where V is constant, ordinates are proportional to *kinematic* pressure.

Drawing tangents to this curve at a and b, we can deduce the values of the indices q and r in the general equation of §§ 35 and 42,—$R = c\, \nu^q\, l^r\, V^r$.

The result given by the curve as drawn is as follows:

$$\textit{Water}: R = c\, \nu^{\cdot 07}\, l^{1\cdot 93}\, V^{1\cdot 93}.$$

$$\textit{Air}: R = c\, \nu^{\cdot 1}\, l^{1\cdot 9}\, V^{1\cdot 9}.$$

These figures must be regarded as a mere illustration. Not only are the data unreliable but there is considerable doubt attaching to the accuracy of a curve drawn through three points only. Observations are wanted made with planes of one standard size and shape, and at a standard velocity in fluids of different ν value, in order that this indirect method of estimating the index values should be really effective.

§ 138. **The Nature of the Pressure Reaction.**—The resistance experienced by an aeroplane in motion is due to the difference of

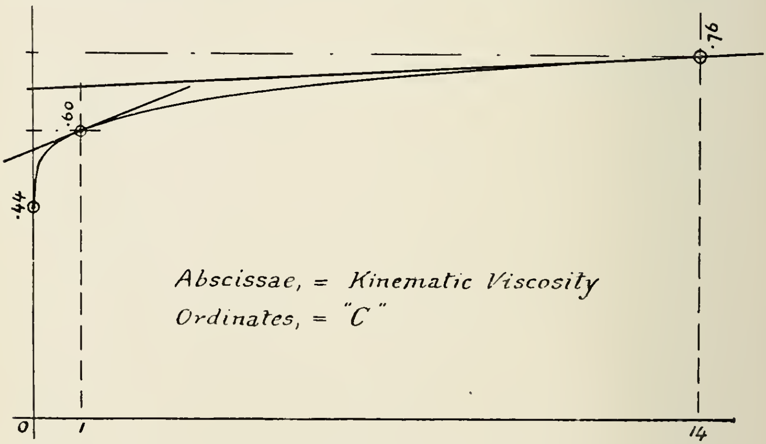

FIG. 87.

pressure between its anterior and posterior faces, and it is the integration of this difference into the area that has hitherto been referred to as the pressure on the plane and denoted by the symbol P.

By the theory of Helmholtz, the pressure difference in a fluid of zero viscosity is entirely due to the excess of pressure on the front face, the "dead-water" being supposed to carry the ordinary hydrostatic pressure of the fluid.

In real fluids there is a viscous drag at the surface of discontinuity, or stratum of turbulence, across which a continuous communication of momentum takes place. This constitutes a force acting rearward on the dead-water as a whole, the reaction of which appears as a region of decreased pressure (a partial vacuum) on the rear face of the plane.

Dines[1] has investigated this point experimentally, and has found that for a one foot square plane in air at 60 *m/h.*, the deficit of pressure on the rear face is approximately one-half the excess pressure on the front of the plane, the measurement being made in the centre of the plane. A similar proportion was found to obtain when the pressure on the mouth of a tube was measured pointed *towards* and *away from* the relative wind direction.

Lord Kelvin has pointed out (*Nature*, p. 597, 1894) that the pressure recorded by Dines in the above experiment, *i.e.*, 1·82 inches of water, corresponds exactly to that given by the Torricellian method, that is to say, that the excess pressure that occurs *at the centre* of a normal plane for any given velocity is that of the corresponding hydrostatic head. This fact is fully consistent with hydrodynamic theory. If the stream lines could be plotted it is evident that at the point on the face of the plane where the stream divides, the velocity of the fluid will be *nil*, therefore by § 82 the pressure at this point will be in excess of that at a distance away by an amount corresponding to the head due to the relative velocity of the fluid.

We have here a definite proof that the Torricellian method is inapplicable in the determination of the constant C, for at every other point on the face of the plane than that at which the stream divides the fluid is possessed of velocity, and consequently its pressure is less than that given by the calculation on the basis in question.

There may be a small departure from the maximum pressure law due to viscosity, but there is every reason to suppose that in fluids of moderate viscosity such error may be ignored; the

[1] "On the Variations of Pressure caused by the Wind blowing across the Mouth of a Tube," Quarterly Journal, Royal Met. Soc. XVI., No. 76, October, 1890.

motion in advance of the plane may be looked upon as irrotational.

The Helmholtz-Kirchhoff result shows that the distribution of pressure over the front of the plane is fairly uniform over the central part, falling off rapidly near the edges; this is evident from the fact that a maximum of ·5 is associated with a mean = ·440 in the case of the infinite lamina. In the case of a plane of compact outline it is probable that the Helmholtz hypothesis would give a considerably lower figure, about ·40 or somewhat less; the maximum, however, will be the same as for the infinite lamina, so that it may be anticipated in this case the pressure will fall off more rapidly towards the periphery.

§ 139. Theoretical Considerations relating to the Shape of the Plane.—The influence of the shape of the plane is most conveniently studied in the two extreme cases to which we have already directed attention, *i.e.*, the compact form (a square or circular disc) and the parallel strip or infinite lamina. The former is a symmetrical case of three-dimensional motion; in the latter the motion takes place in two dimensions only.

It has been sometimes suggested that since the pressure increases with the relative periphery, the pressure is greatest in the peripheral regions; we have already seen that such is not the case. The true reason is to be found in a complication of causes.

(1) The congestion of fluid that gives rise to the pressure region is less when the fluid can escape laterally in *two dimensions* than when its "spread" is *confined to one dimension.*

(2) The "spurting" of the lines of flow past the edges of the plane will be greater when the access of the fluid to the "hinterland" is the more complete. Thus in an infinite strip of width = b the layers of fluid adjacent to the face of the plane are fed by a much greater stream area than in the case, say, of a circular disc of which b is the diameter, and the spurting past the edge of the plane will be correspondingly the more vigorous;

this is represented diagrammatically in Fig. 88. It is evident that the plane that causes the greater displacement of the lines of flow will experience the greater pressure.

(3) The viscous drag on the dead water will be greater when the periphery is greater. Thus, the pressure on the rear face of the plane will be less, that is, the vacuum will be greater for planes of elongate or erratic form.

§ 140. Comparison with Efflux Phenomena.—An analogous case illustrating the foregoing principles is to be found in the efflux of fluids under pressure (§ § 95—96). In the case of a jet issuing from a simple circular orifice we have a case of three-dimensional motion, and as the flow takes place *inwardly* the layers of fluid in the vicinity of the orifice will be fed by a greater stream area than would be the case if the orifice had the form of a slit and the motion in two dimensions; the "spurting" at the edge will therefore be more vigorous and the contraction of the jet will be greater. This is found experimentally to be the case, the coefficient of contraction being usually taken, for the circular aperture, as from ·615 to ·620, whereas for a slit aperture it is found to be about ·635. On the principles discussed in § 95, the greater the jet contraction the less the pressure is relieved on the wall of the vessel in the vicinity of the orifice.

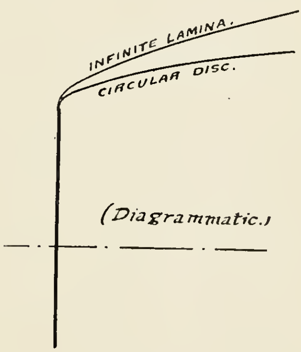

Fig. 88.

We may follow the comparison further. In the case of the Borda nozzle the access of the fluid to the jet is improved by the arrangement of an inwardly projecting "lip" so that the pressure on the wall of the vessel undergoes next to no reduction, and the coefficient of contraction becomes (theoretically) = ·5 or by experiment ·515.

By similarly fitting a lip or a projection to the edge of the normal plane, opposed to the relative direction of the wind, its pressure constant can be considerably increased. If the lip be of sufficient height to render motions of the fluid adjacent to the plane itself very small, so that the square of such velocity as it may possess may be everywhere negligible, then the pressure on the face of the plane will, on the hydrodynamic principle already cited (§ 138), be everywhere that due to the Torricellian head, and the pressure constant will be ·5; it would appear to be impossible for it to rise above this value.

In viscous fluids there would be doubtless some departure from strict theory, owing to the fact that the fluid in advance of the plane has rotation impressed upon it by viscous stress, and the hydrodynamic principle assumes irrotation; in ordinary fluids the error due to this cause should not be great. Beyond this there is the separate phenomenon of the suction on the back of the plane, which may be regarded as supplying an added constant, the sum of this and the pressure constant making the C of the equation.

§ 141. The Quantitative Effect of a Projecting Lip.—For planes of compact outline, Dines obtained the following results :—

Plane 1 foot diameter, circular.

Projection of lip or rim.	Percentage increase.
$\frac{1}{8}$ inch	6 per cent.
$\frac{3}{8}$,,	10 ,,
$\frac{5}{8}$,,	14 ,,

We have stated that the *probable* value of the pressure constant on the Helmholtz basis for a plane of compact outline is about ·40 or somewhat less; this would give a possible augmentation of 25 per cent. or somewhat more (the limit being ·5 according to the preceding article); but Dines' result is the percentage on the whole constant and requires to be multiplied by 66/40, so that his figure for a $\frac{5}{8}$ inch rim becomes 23 per cent. This result is in harmony with the theory, but would seem to point to the probability of a lower value than ·40 for the Helmholtz constant, in view of the probability of higher resistances being experienced with greater depths of rim. It is worthy of remark that Dines obtained, for a hemispherical cup, pressures about 16 per cent. greater than for a plane circular disc.

In general, if we neglect the influence of rotational motion *within* the stream, let, as before, C be the experimentally ascertained constant for any plane, and c_1 the pressure constant on the Helmholtz hypothesis, and let $C + n\,C$ be the total augmented pressure for the same plane fitted with a deep rim, we shall have the relation :—

$$c_1 = \cdot 50 - n\,C \quad \text{or,} \quad n\,C = \cdot 50 - c_1.$$

If we apply this to the two-dimensional case of the infinite strip, we have Kirchhoff's determination of the Helmholtz constant $c_1 = \cdot 440$, and Dines' experimental result $C = \cdot 76$, so that—

$$\cdot 76\,n = \cdot 50 - \cdot 44 \quad \text{or,} \quad n = \cdot 08.$$

That is to say, the maximum possible addition to the pressure is, in this case, about 8 per cent.

Dealing with this problem in an analytical investigation, Love[1] has shown that if ϵ be the ratio of height of lips to breadth of plane, the pressure will be increased *approximately* by an amount $= \frac{6}{5}\sqrt{\epsilon}$ of that for the same plane without the lips. It is evident that this expression only holds good for small or moderate values of ϵ, for the limiting value would otherwise be exceeded. This would occur when $\sqrt{\epsilon} = \frac{5}{6} \times \cdot 08 = \cdot 0666$ or, $\epsilon = \cdot 258$, so that Love's approximate equation will be perceptibly in error for some considerably smaller value.

§ 142. Planes of Intermediate Proportion.—We have so far dealt with the two extreme cases of contour form typical of two-

[1] "Theory of Discontinuous Fluid Motion," Proc. Camb. Phil Soc., VII., 1891.

dimensional and three-dimensional motion, the infinite strip, and the plane of compact outline.

Our knowledge of other forms is at present somewhat limited. It may be fairly assumed that, just as the value of the constant is, within the limits of observation, the same for the circular as for the square form, so for an ellipse or other tolerably regular elongate form it will be the same as for a rectangle of like proportions. We will therefore confine our attention in the present section to rectangular planes of different proportions.

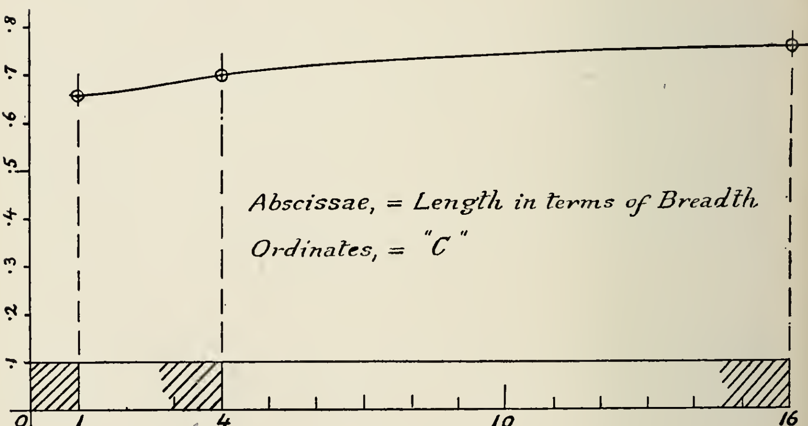

Fig. 89.

We have to rely chiefly on the observations of Dines for data. Fig. 89 gives the value of C plotted as a function of the length of the plane in terms of its breadth, the form of the plane being represented graphically by the shaded area. The small circles denote the observation data on which the curve is based. The curve is not carried beyond the ordinate proper to the square plane, as it obviously repeats itself, the corresponding abscissae being in arithmetical and harmonical progression respectively.

For planes of highly irregular form no definite rules can be laid down. An assumption that such planes are built up of simpler components will sometimes enable the value of C to be assessed; but as the whole range of C values lies almost within the admittedly possible allowance for experimental error, our want of knowledge on this point is not so serious as might otherwise appear.

§ 143. **Perforated Plates.**—Dines has investigated the effect of perforations as affecting the resistance of the normal plane. In one case a plane one foot square was taken and eight circular holes, each one square inch area, were punched, as illustrated in Fig. 90; no difference of pressure could be detected whether any or all of the holes were covered or open. Mr. Dines remarks: "The eight holes together take away more than 5 per cent. of the plate, yet a difference of 1 per cent. in the pressure, had it existed, would certainly have been apparent." Further experiments were made with two kinds of perforated zinc, the one sample, holes ·08 inch diameter 77 per square inch, having only 61 per cent. of the total area, was found to give 91 per cent. of the total pressure; another sample with perforations ·22 inch diameter, 11 or 12 per square inch, possessing only about 56 per cent. of the total area, gave 80 per cent. of the pressure on a solid plate.

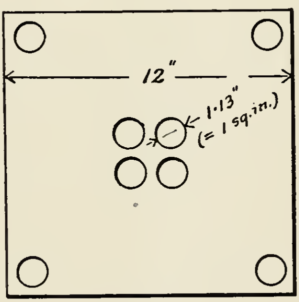

Fig. 90.

The curve given in Fig. 91 is deduced principally from Dines' experiments. Abscissæ give percentage area removed, and ordinates show the corresponding pressure as a percentage of that on the same area intact. It is supposed that when the percentage of area remaining becomes small, the perforations are of square form as indicated.

The anomalous behaviour of the perforated plate is perfectly explicable on theoretical grounds.

The region in the rear of the plane is occupied by the turbulent "dead-water" at a pressure below that of the undisturbed fluid. When a hole is made in the plane, the air flows through from the front to the rear under the influence of the difference of pressure between its two faces. The stream of air finding its way through the perforation carries with it an amount of momentum per second, equal to the force of which the plane is relieved. If there were a conduit to carry this efflux air away without interfering with the dead-water, then the plane would

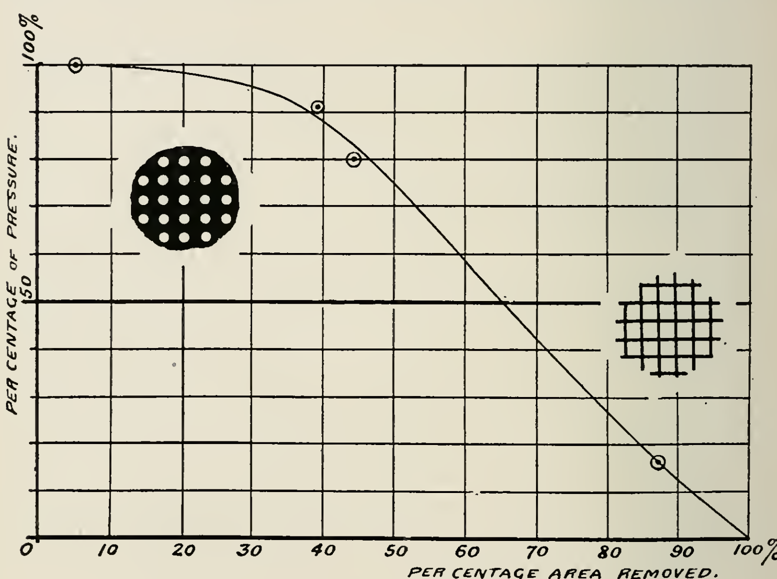

FIG. 91.

show itself relieved of pressure to the extent of the area taken away together with an addendum due to the fall of pressure adjacent to the perforation, in accordance with the well-known principles of efflux theory. But there is no conduit to carry away the efflux air, which consequently passes into and becomes mixed up with the turbulent dead-water; and the efflux air carries with it its momentum, which is communicated to the dead-water, and momentum communicated to the dead-water appears as negative pressure on the rear face of the plane, since it is the plane itself that prevents the dead-water from being washed bodily away. Consequently the vacuum on the rear of the plane is increased to just the same extent as the pressure on the front is diminished, both quantities being measured by their integration over the respective faces of the plane; that is to say, the existence of a perforation has no influence on the total reaction on the plane.

When perforations are made of great size in proportion to the dimensions of the plane, we can conceive of the efflux stream passing *en masse* through the dead-water without parting with the whole of its momentum, so that in such a case the plane will be relieved of a portion of its resistance. The same may be supposed to happen if the perforations become sufficiently numerous.

Note.—In the present chapter the discussion is based principally on the result of Mr. Dines' investigations, the value of C for the plane of compact form being taken at ·66. The author has been influenced in this partly by the fact that in all probability Dines' results are nearer the truth than those of Professor Langley, but more particularly by the consideration that when instituting a comparison it is safer to confine one's attention to the work of a single investigator, and Langley's experiments with the normal plane were not carried far enough to give the information required.

For general employment in the subsequent volume ("Aerodromics") the value of C is taken as ·7, which is the result given by Langley for a plane of square form and corresponds with the result given by Dines for a plane 4×1.

In adopting this value it has been borne in mind that it is desirable to have a general average figure that can be used with safety without specifying the exact form of the plane, and, taking C as ·7, it will not matter seriously whether Langley's or Dines' result should ultimately prove to be the nearer to the truth.

CHAPTER VI.

THE INCLINED AEROPLANE.

§ 144. Introductory. Present State of Knowledge.—The problem presented by the inclined aeroplane is of very great complexity, and no general solution has at present been found. Our knowledge of the behaviour of the plane inclined to its direction of motion is in the main confined to the immediate results of experiment, extended it may be by the drawing of smooth curves through the observed points plotted on a co-ordinate chart. In certain extreme cases theoretical solutions have been found, and in other instances empirical formulæ have been proposed, in fairly close agreement with the results on which they are based.

In addition to the considerations that weigh in the case of the normal plane, we have now not only to deal with some unknown law correlating pressure and angle, but we have also to take account of the remarkable effects due to the influence of *aspect*.[1]

The early writers on fluid dynamics did not draw a proper distinction between an *aeroplane* and the *surface of a solid* of similar form, such for example as the wall or roof of a building; this has resulted from a too literal application of the impact theory of Newton. The pressure on a circumscribed area of the surface of a solid cannot be given by any formula, or, in fact, at all, unless the form of the remainder of the solid be known; any equation or theory that attempts to give a solution for the individual elements of the surface of a body independently of its

[1] A word due, in its present usage, to Langley. By *aspect* is meant the arrangement of the plan-form of an aeroplane, or other aerofoil, in relation to the direction of flight.

whole form, is of necessity unreliable and in general entirely misleading.

An aeroplane may be regarded as the special case of a body whose whole form is defined by the shape of its face in presentation, and consequently in stated *aspect* its pressure reaction can be expressed as a function of its angle and velocity.

§ 145. The Sine² Law of Newton.—The difference between the behaviour of a real fluid and the Newtonian medium, sufficiently evident in the case of the normal plane, is further accentuated when the effect of inclining the plane is taken into account.

According to the hypothesis of the Newtonian medium the

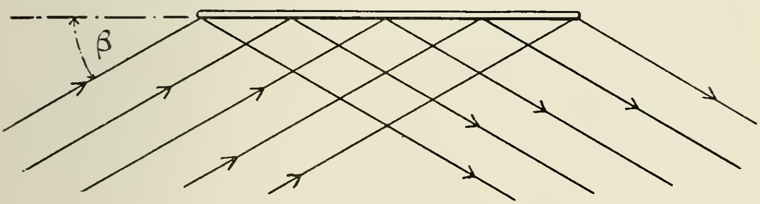

Fig. 92.

pressure is due to the impact of the particles of which the medium is composed. In the present case it is simplest to presume, in the first instance, that the plane and particles are perfectly elastic. Let Fig. 92 represent a plane the pressure on which is due to the momentum communicated by a Newtonian medium, whose relative path is that indicated by the arrows making an angle β with the plane itself. Then if A be the area of the plane, and V the velocity of the "medium" whose density is ρ, the total momentum of the stream per second $= \rho\ A\ V^2\ sin\ \beta$, and component normal to plane, $= \rho\ A\ V^2\ sin^2\ \beta$, and on the assumption of perfect elasticity the total momentum communicated per second is :—$2\ \rho\ A\ V^2\ sin^2\ \beta$, or, if P_β = pressure per unit area on the plane we have—

$$P_\beta = 2\ \rho\ V^2\ sin^2\ \beta. \qquad (1)$$

But for the normal plane, denoting the pressure by the symbol P_{90}, we know that for the conditions of the present hypothesis—

$$P_{90} = 2 \rho V^2 \quad (\S\ 136),$$

or we have—

$$P_\beta = P_{90} \times sin^2 \beta. \qquad (2)$$

If we modify the hypothesis to the extent of supposing the plane *inelastic* both P_β and P_{90} are diminished in the ratio 2 : 1,

or, $$P_\beta = \rho V^2 sin^2 \beta,$$

and $$P_{90} = \rho V^2,$$

∴ the relation $$P_\beta = P_{90} \times sin^2 \beta$$ still holds good.

We may view this problem in another light with the same result. If we regard the motion of the plane as compounded of its edgewise and normal components, then the former can be neglected since it does not involve any reaction on the plane. Now if V_1 be the value of the normal component, the mass dealt with per second is $\rho A V_1$ and the momentum per second is $\rho A V_1^2$, or (on the elastic hypothesis),

$$P_\beta = 2 \rho V_1^2 \qquad (3)$$

which is the same as (1), for $V_1^2 = V^2 sin^2 \beta$.

So that the pressure in the Newtonian medium is independent of the edgewise component of motion, and is the same as for a normal plane of velocity equal to the normal component of the actual motion.

An important consequence of this is that if we had to do with a Newtonian medium, or if a real fluid behaved as such, then the time of falling of a horizontal plane would be independent of any horizontal motion impressed upon it. The "falling plane," therefore, becomes the *experimentum crucis* in respect of the "sine square" law.

§ 146. The Sine² Law not in Harmony with Experience.—It has long been known that in actual fluids the sine square law does not hold good. Probably the first experimenter to ascertain this fact was Vince in the year 1797 (Phil. Trans., 1798); later we find an explicit statement by Robinson (System of Mechanical

Philosophy, 1822), that: "The resistances do by no means vary in the ratio of the squares of the sines of the angle of incidence; and for small angles the resistances are more nearly proportional to the sines than to their squares." This is a very important statement, from which a vast amount of inference may be drawn; it has been fully justified by the subsequent work of Wenham, Dines, Langley, and others.

The most direct disproof of the Newtonian law is to be found in experiments with the *falling plane*. It is found that if a horizontal plane, suitably mounted in vertical guides, be allowed to fall freely, the time of fall may be increased almost indefinitely by imparting to it a simultaneous horizontal motion. This was pointed out by Wenham in the year 1866, and has more recently been brought into prominence by the experiments of the late Professor Langley. Langley employed an appliance which he termed a "plane dropper," mounted upon the arm of his "whirling table" (§ 233), for making his determinations. It is of interest to note that although Langley took occasion more than once to comment upon the defects of the Newtonian law, as a deduction from his other experiments, he did not apparently appreciate that the falling plane really constitutes a *direct* disproof.

§ 147. The Square Plane.—The nature of the Newtonian discrepancy and the extent of agreement between the work of different investigators may be exemplified in the case of the plane of square form.

The square plane may be taken as the type of greatest simplicity which includes generally planes of *square proportion;* such planes are not affected seriously by considerations of "aspect," although doubtless a square plane will not give exactly the same results in diagonal as in square presentation. The latter is always assumed in the absence of an explicit statement to the contrary.

In Fig. 93, in which ordinates represent the relative pressure on the plane for all angles from 0 degree to 90 degrees, we have curves plotted as follows:—

(*A*) The Newtonian or sine² law— $P_\beta = P_{90} \sin^2 \beta$;

(*B*) According to an empirical formula proposed by Duchemin

$$P_\beta = P_{90} \frac{2 \sin \beta}{1 + \sin^2 \beta};$$

(*C*) From the experiments of Langley;

(*D*) From the experiments of Dines.

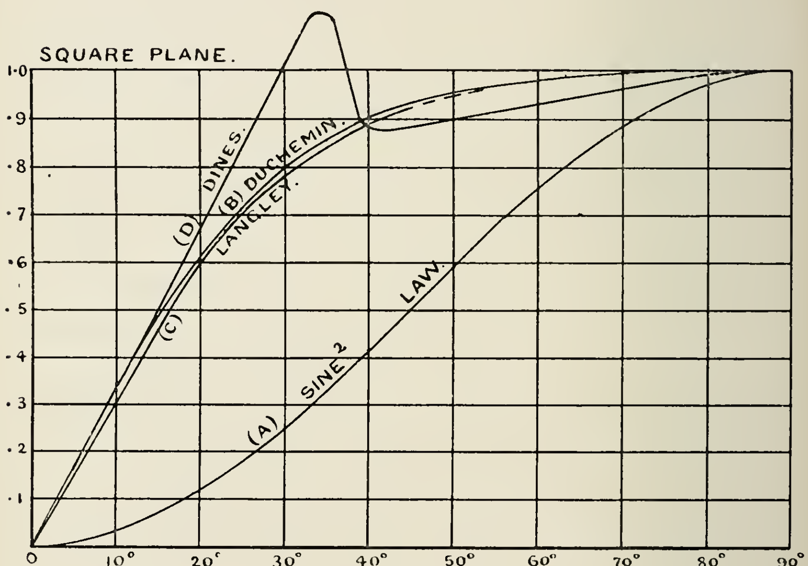

FIG. 93.

It will be noticed that the Newtonian curve does not accord with any of the experimentally ascertained curves, which latter do not even agree very closely amongst themselves. Perhaps the most salient facts connected with these curves are the close agreement between Langley and Duchemin (an agreement pointed out by Langley in his Memoir), and the remarkable disagreement in the curve of Dines, characterised by an erratic "kick up," a maximum being recorded at or about 55 degrees angle, at which point the *pressure is actually greater than when the plane* *is normal.* This is one of the many experimental disagreements that at present are far too common in aerodynamics.

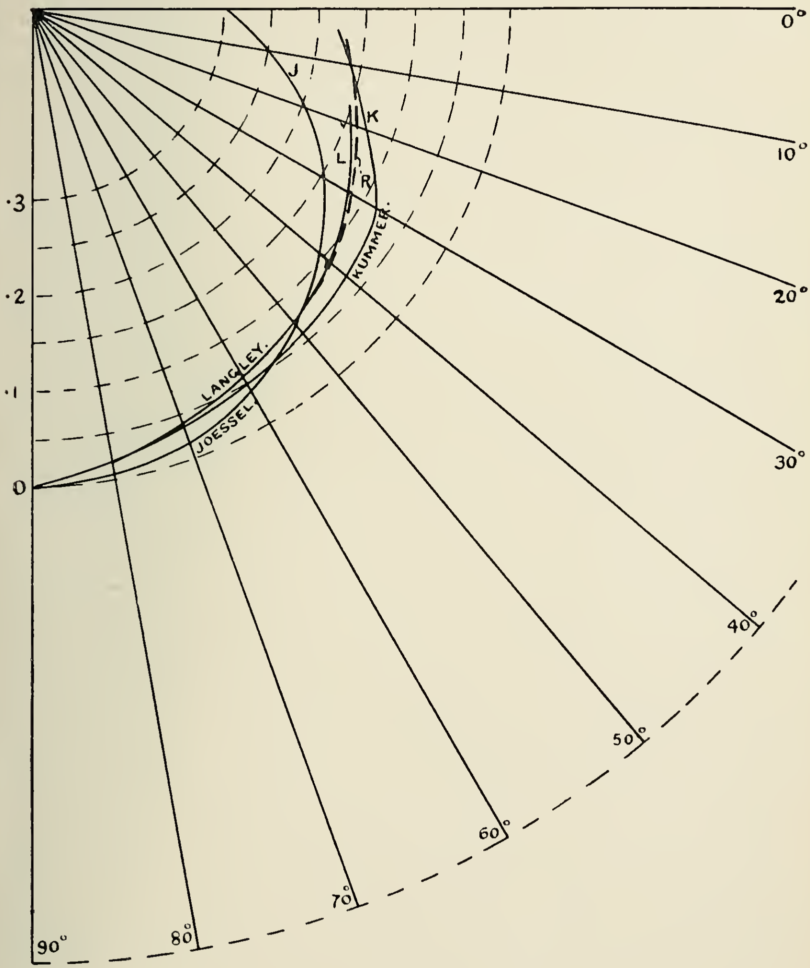

FIG. 94.

§ 148. The Square Plane.—Centre of Pressure.—According to the Newtonian hypothesis, the centre of pressure on an inclined plane should be coincident with the geometric centre. In real fluids this is found not to be the case.

This point has been fully investigated, so far as the square plane is concerned, by Joessel, Kummer, and Langley. It is found that for the normal plane the geometric and pressure centres are coincident, but that as the plane is inclined the latter is displaced towards the leading edge, the displacement of the centre of pressure increasing as the angle made by the plane to its line of flight becomes less and less. This is shown in the form of a diagram in Fig. 94, in which it is supposed that the plane is swung through a quadrant, from zero to 90 degrees, the *locus* of its centre of pressure, as determined by the different observers, being indicated in the figure, in which also are given the position of the plane at every 10 degrees angle, and one-tenth divisions from which the position of the centre of pressure may be read in terms of the width of the plane.

The general character of the curves of the square plane, both as to magnitude and location of pressure, are shared to a greater or less extent by planes of other proportions.

§ 149. Plausibility of the Sine^2 Law.—The general acceptance of the experimental fact that the sine^2 law is in error, has without doubt been delayed by the very plausibility of the law itself.

If we suppose (as is quite customary in dealing with physical problems) that the diagonal motion of the plane is compounded of its edgewise and normal components, then, as in the previous discussion (§ 145), we may, neglecting skin friction, regard the former as of no influence and the pressure as due entirely to the normal component. In greater detail, if we suppose the motion of the plane to take place in steps, *i.e.*, alternate edgewise and normal movements, and if we assume the former to take place with infinite rapidity, and the steps to become infinitely numerous, then it would appear that the pressure due to the inclined motion has been demonstrated to be, in effect, exactly that due to the normal component of the whole motion.

The above reasoning is manifestly in error, since the result does not accord with experience. The fallacy has been pointed out by Lord Rayleigh, whose explanation is substantially as follows:—

When the plane undergoes the edgewise component of its motion, it abandons air which has been set in motion (normally) at its trailing edge, and embraces air that has not been set in motion at its leading edge. This exchange obviously results in an augmentation in its resistance. This reasoning applied to the "step by step" motion evidently continues to apply when the steps become infinitely small and the motion continuous, consequently the pressure will be greater than that due to the normal component alone, as is found experimentally to be the case.

It appears to the author that the augmentation of pressure will be greater than might be supposed from the foregoing reasoning, for the abandoned air, having motion in the same direction as the plane, will impede the flow of air round the following edge and so maintain a greater pressure difference between its two faces; likewise the new air seized by the advancing edge being already in circulation round that edge has a higher velocity relatively to the plane than the normal component of motion, so that the pressure it will develop will be greater than if it had been merely new air coming into the grasp of the plane. We are now evidently touching on the subject of Chap. IV., and dealing with the pressure due to the cyclic disturbance; this aspect of the subject will be resumed later.

§ 150. The Sine-Squared Law Applicable in a Particular Case.—It is evident from the foregoing reasoning that planes of different *aspect ratio*[1] will have their normal pressure components augmented to different degrees, inasmuch as the relative extent of their leading and trailing edges differ.

If we consider the case of a plane of extreme proportion it is

[1] A term used in the present work to denote the lateral dimension of an aeroplane, or other aerofoil, in terms of its fore and aft dimension, denoted by the symbol **n**.

obvious that in *apteroid aspect*[1] the augmentation will be very small indeed, and if we go so far as to suppose the plane of infinite length, then the augmentation vanishes.

Thus the infinite parallel lamina, in apteroid aspect, affords the case of a plane that will conform to the sine^2 law, and the pressure on its faces will be given by the expression: $P_\beta = P_{90} \times sin^2\,\beta$, or in full: $P_\beta = C\,\rho\,V^2\,sin^2\,\beta$ where C is the constant of the normal plane, or in absolute units for a plane of the form under discussion in air, $C = \cdot 76$ or $\cdot 78$ (about).

The above result becomes self-evident from the point of view of relative motion. The conditions of the problem will be fully represented if we suppose an infinite parallel lamina in normal presentation to slide along in the direction of its own length. It is evident that such sliding motion, presuming no skin-friction, can have no effect whatever upon the pressure reaction, and therefore by § 145, the sine^2 law holds good.

We might go so far as to suppose the above experiment to be tried on a "whirling table" (Chap. X.), the plane being extended to form a *complete ring* bounded by two concentric circles. Assuming the method to be that of the *falling plane*, it is evident that the time of fall of such a ring will be substantially independent of its velocity of rotation.

§ 151. Planes in Apteroid Aspect (Experimental).—In Fig. 95 we have plotted *to a common maximum value :* (A) the curve of sine^2 as deduced in the preceding article for the special case where the plane extends to infinity; (B) the Duchemin curve for the square plane, the Dines curve also being shown dotted; (E) curve as plotted by Langley for plane, 6 inches by 24 inches; (F) curve as plotted by Dines for plane, 3 inches by 48 inches. If, as there is every reason to suppose, the normal pressure is a *continuous function* of the aspect ratio of the plane, then as we suppose the latter to undergo variation from the square to the infinite

[1] With the greater dimension arranged in the direction of flight, in contradistinction to *pterygoid.*

lamina the curve will pass gradually from the form given by (*B*) to that given by (*A*), (*E*) and (*F*) being intermediate stages, and we may expect that the whole intermediate series will be in most part included within the area between the curves *A* and *B*, and in their character the intermediate curves will form a homogeneous series; thus a few accurate plottings from planes of

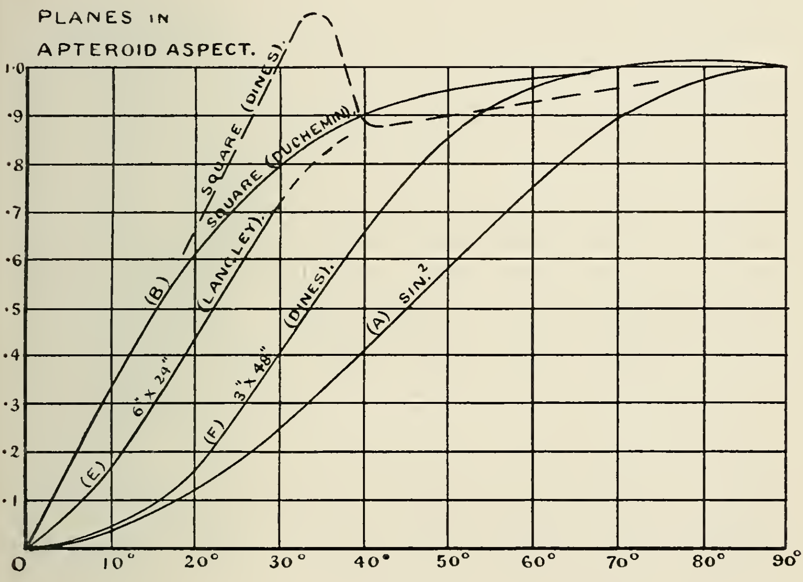

FIG. 95.

known proportions would enable the curve to be drawn for any intermediate plane with a reasonable degree of certainty

The curves as plotted in Fig. 95 are to some extent misleading, each curve being plotted in terms of the common maximum ordinate. In Fig. 96 the necessary correction is made to reduce the curves to a *common scale*, the maximum values being assigned proper to each particular proportion of plane in accordance with Fig. 89 (Chap. V.).

In Fig. 96 abscissæ represent angles of inclination as before, and ordinates give the values of the constant *C* generalised so that (in absolute units) $P = C \rho V^2$ where P is the normal pressure on the plane for any angle.

There is some doubt as to the correct plotting of the Langley curves owing to the fact that this observer was unaware of the variation to which the C of the normal plane equation is subject, as dependent upon the shape of the plane.

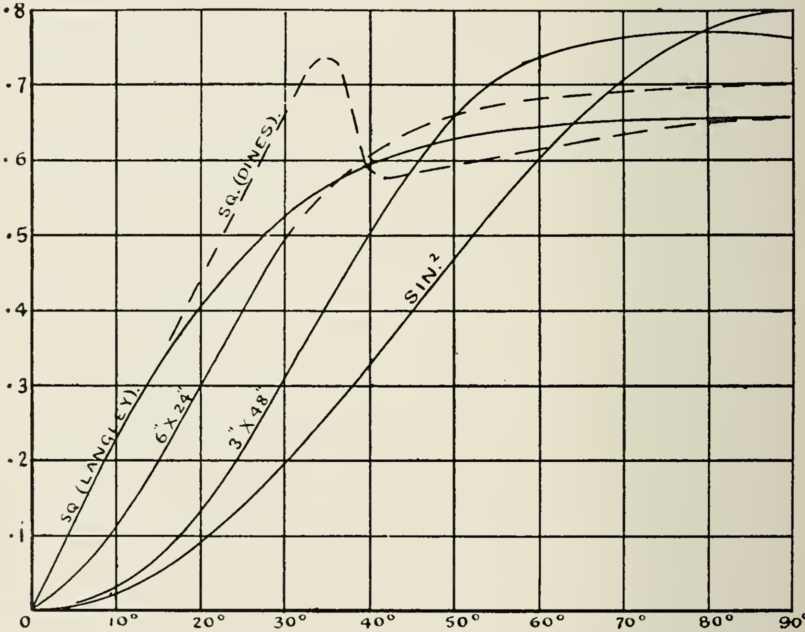

FIG. 96.

§ 152. The Infinite Lamina in Pterygoid[1] Aspect.—The case of the inclined infinite lamina in pterygoid aspect has been examined by Kirchhoff and Rayleigh on the Helmholtz hypothesis. According to this investigation the pressure is given by the equation: $P = \frac{\pi \sin \beta}{4 + \pi \sin \beta} \rho V^2$ from which the curve (Fig. 97)

[1] With the lesser dimension in the direction of flight, as in the wing plan-form of birds.

is calculated and plotted.[1] The ordinate scale is given in terms of maximum value = 1, and in terms of C, value in which case becomes ·440.

According to the theory advanced by the author (Chap. IV.), the case now under discussion is indeterminate; the reaction on the plane is a function of the strength of the cyclic motion and its velocity of translation, and is not dependent upon the angle in the particular case of the plane of infinite lateral extent. If

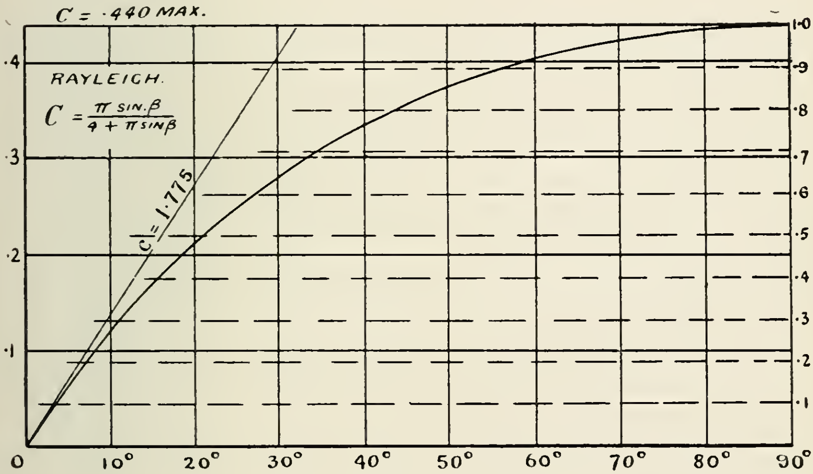

Fig. 97.

the plane, although of great aspect ratio, be of finite length, then a dispersal of the energy of the cyclic motion will take place, and in order that a steady state should exist this energy must be continuously renewed by the work done in proportion, which requires some specific angle in order that a stated load shall be sustained.

In a real fluid it is evident that the type of motion depicted in Figs. 71 and 75 could not exist *in toto*; the abruptness of the motion round the sharp edges of the plane would give rise to discontinuity. From our knowledge of problems of this kind we

[1] See also § 97.

may predict the general character of the resulting flow (Fig. 98) (*a*), which in all probability would be accounted for by the Kirchhoff-Rayleigh analysis. There is, however, some probability that when the inclination of the plane is small the viscous drag ejects the dead-water from the region above the plane, as in the case of the stream-line body, so that the motion will be approximately as represented in Fig. 98 (*b*), in which a small remnant of the dead-water alone remains immediately above the front edge of the plane. The resulting type of motion from a hydrodynamic standpoint is somewhat obscure; that a cyclic component exists there can be no doubt, but it is difficult to frame a *régime*

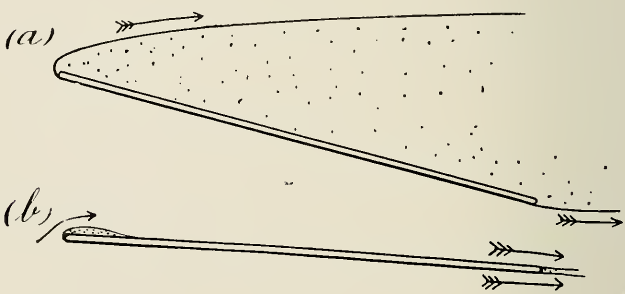

Fig. 98.

which is in strict accord with hydrodynamic principles. It is possible that the surface of junction of the two streams, when they meet at the after edge of the plane, contains *rotation*, there being a finite difference between the velocities, and that this region of rotation modifies the lines of flow of the cyclic system in a manner that remains for future investigation.

If the author's theory is correct in its present application, the Kirchhoff-Rayleigh result will break down for small angles, in the direction of showing too low a reaction; for it is evident that the arrangement of flow (*b*) (Fig. 98) will result in a greater downward velocity being given to the air than in case (*a*). Experimental evidence on this point is at present inconclusive.

§ 153. Planes in Pterygoid Aspect (Experimental).—The experimental information at present available relating to planes in pterygoid aspect is very unsatisfactory and conflicting.

In Fig. 99 we have a plotting given by Langley in the case of a plane 30 inches by 4·8 inches, with the curve for a square plane given for comparison. It was pointed out by Langley that the pressure on a plane in pterygoid aspect is greater *for small angles* the more extreme the proportion, but that this rule does

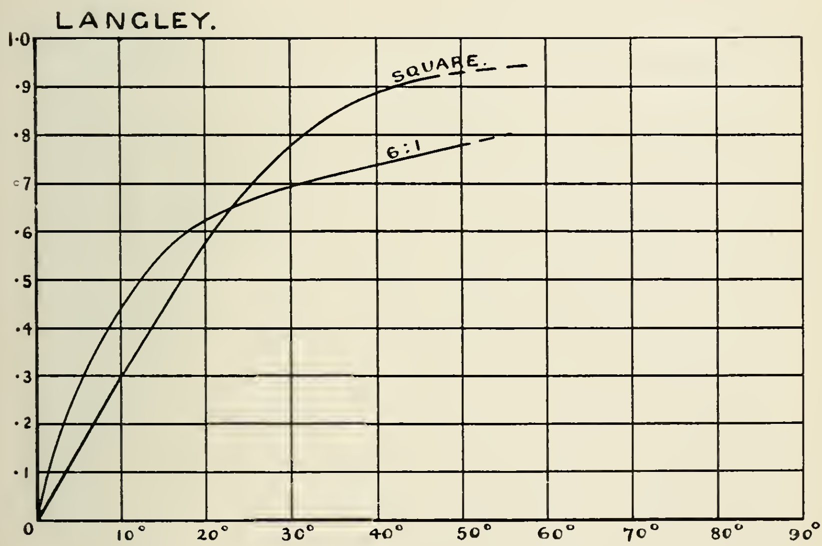

Fig. 99.

not hold good when in the comparison of any two planes the angle exceeds a certain critical value. Thus in the figure the 30 inch by 4·8 inch curve crosses the 12 inch by 12 inch curve at an angle of about 23 degrees.

Curves as plotted by Langley are not fully comparable, in view of the fact, discovered by Dines, that the *shape* of the normal plane affects its pressure.

In his investigations on the influence of aspect, Dines has failed to show any trace of the "reversal," or crossing, of the curves, so clearly brought out by the experiments of Langley. It is possible that the form of the so-called "planes" employed by Dines is responsible for much of the disagreement. Dines employed slabs of triangular section (Fig. 100) (*a*), whereas Langley adopted a flat section (*b*), his "planes" having square edges, and being of about one-eighth inch thickness.

In Fig. 101 the curves are plotted *B*, *B* for a "plane" of square form, and (*F*) for one measuring 48 inches by 3 inches in pterygoid aspect, from Dines' paper (Proc. Royal Soc., Vol. 48). These curves incidentally cross one another, but there is nothing resembling Langley's reversal.

In experimental aerodynamics we are used to encountering discrepancies of various kinds, but a disagreement of the present

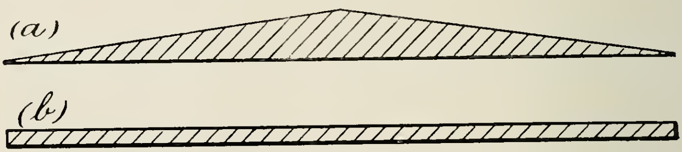

Fig. 100.

extent is most unsatisfactory. On the whole, for planes at small and moderate angles, the author is disposed to accept Langley's data as the more reliable.

§ 154. Superposed Planes.—The effect of the proximity of one aeroplane to another has been investigated experimentally by Langley. In a series of experiments carried out by the aid of his whirling table and "*Plane-Dropper*," Langley showed that two parallel planes, one above the other, will, at a given angle, support as great a load as if they were entirely independent, so long as they are separated by a certain minimum distance. In the actual experiments two pairs of planes, each 15 inches by 4 inches, were employed in pterygoid aspect (Fig. 102); it was found that so long as the angle did not exceed a certain maximum value a separation of four nches (*i.e.* equal to the fore and aft dimension of the planes) was sufficient to prevent any sensible interference between the two pairs, so that each would carry the same load as if the other were absent. Trials with the planes two inches apart showed a falling-off of about 15 per cent. of the total load.

It would appear that when the inclination of the plane exceeds five or six degrees, some interference is felt even at four inches

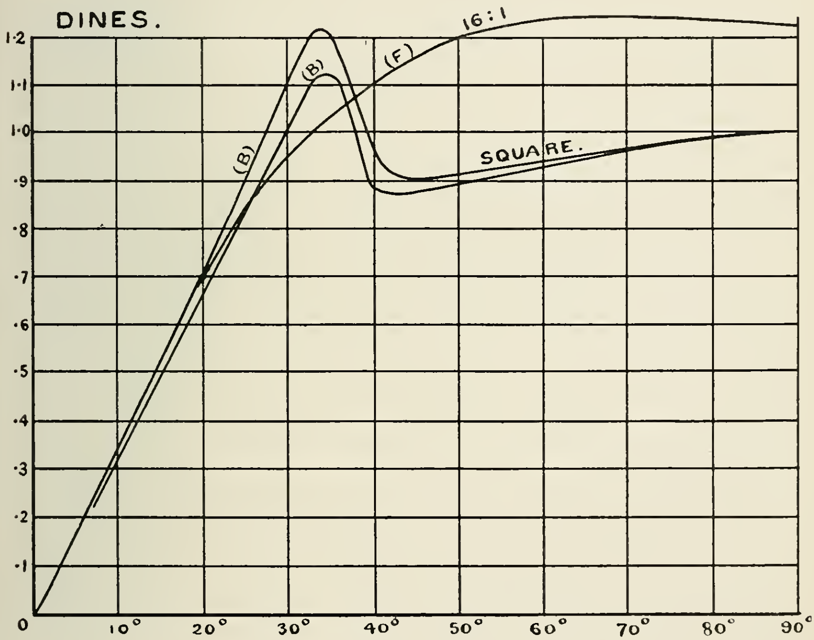

Fig. 101.

separation. This is doubtless to be attributed to some change in the type of fluid motion, such as that suggested in Fig. 98; on the other hand, no increase in the velocity, and therefore diminution of the angle, is found to prevent the interference taking place when the distance apart is reduced to two inches.

Referring to these experiments Langley says:—"The most general, and perhaps the most important, conclusion to be drawn from them, appears to be that the air is sensibly disturbed under the advancing plane for only a very slight depth; so that for the planes four inches apart, at the average speeds, the stratum of air disturbed during its passage over it is, at any rate, less than four inches thick. In other words, the plane is sustained by the compression and elasticity of an air layer not deeper than this, which we may treat for all our present purposes as resting on a solid support less than four inches below the plane." "(The reader is again reminded that this sustenance is also partly due to the action of the air above the plane.)"

In the author's opinion the whole of this inference is unsound. Professor Langley appears to have overlooked the possibility of

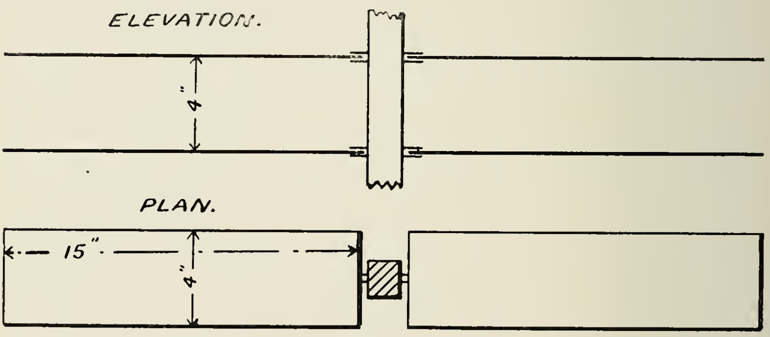

FIG. 102.

a *superposition of the two systems of flow*, such as is plotted for the Eulerian fluid in Fig. 73, Chap. IV. It is evident that such a superposition is not only possible but highly probable, each plane affecting the other profoundly so far as the actual stream-line system is concerned; but the combined supporting power of the planes, that is, the sum of the two systems, being substantially unaffected. Fig. 73 illustrates, from the case of the perfect fluid, the manner in which the two cyclic systems proper to the two planes may react on one another.

Beyond this it is evident that the speed of propagation of the compression and rarefaction within the fluid will be equal, or approximately equal, to that of sound, so that if this were really the determining factor as affecting the layer of fluid involved, the latter would be much greater than the observed four inches; it would, in fact, at the average velocity employed, amount to something like eight feet in either direction, that is to say, some sixteen feet in all.

The employment of a series of superposed members for the support of a load in flight was not new at the date of Langley's experiments. This system appears to have been well known to, if not actually employed by, Horatio Phillips, being foreshadowed in his specification of 1884, and further in 20,435 of 1890, and very thoroughly developed in his captive machine at Harrow about the same date. The supporting members adopted by Phillips were rightly of curvilinear section (see Fig. 60), but the critical distance of separation is evidently much the same for such a form as for a *plane;* at least Phillips appears to have independently adopted for his aerofoil spacing substantially the proportions subsequently proved by Langley to be admissible for the aeroplane.

§ 155. The Centre of Pressure as affected by Aspect.—The general behaviour of the centre of pressure as a function of the angle has been discussed in respect of the square plane in § 148. It remains for us to examine the subject in its relation to *aspect.*

So far as the author is aware, the only experimental determinations other than for the square plane are those of Kummer (Berlin, *Akad. Abhandlungen*, 1875—6), from which it appears that the displacement of the centre of pressure from its normal position is less in planes in apteroid aspect than in the square plane, and is greater in planes in pterygoid aspect. This is substantially what might be anticipated, for in the case of the infinite lamina in apteroid aspect the pressure distribution along its length is uniform, so that the centre of pressure for a *very long plane* will be sensibly undisturbed by its change of angle. On the other hand, in planes in pterygoid aspect the cyclic motion results in an increased pressure region under the leading edge, and in a partial vacuum in the region above. If the cyclic motion were perfect, as in Figs. 71, 72, 73, &c. (Chap. IV.), the motion of the fluid would be symmetrical, and the centre of pressure would not suffer displacement; owing, however, to the imperfection of real fluids, the pressure on the region of the following edge is not materialised, the motion becoming discontinuous, as depicted in Fig. 98, so that the centre of pressure is situated towards the forward edge of the plane.

Langley has observed that when the angle of flight exceeds a critical value, the displacement of the centre of pressure is greater for planes in apteroid than in pterygoid aspect, a reversal taking place similar to that discovered by him in the case of the total pressure reaction.

The position of the centre of pressure as a function of the inclination is of most interest in the case of planes of extreme proportion in pterygoid aspect. Under these conditions experiment is most difficult; no reliable data are at present available.

Lord Rayleigh has given the theoretical solution in the case of the *infinite lamina* in pterygoid aspect, on the Helmholtz hypothesis (§ 97). It is a curious fact that, when plotted, Rayleigh's curve is almost identical with that based on Langley's observations *for the square plane*, the departure only becoming noticeable at small angles; see Fig. 94 (L = Langley, R = Rayleigh).

§ 156. **Resolution of Forces.**—It is one of the advantages of the aeroplane as a medium of experiment that, if we *neglect any tangential forces* acting on its surfaces, the total pressure reaction, the resistance in the line of flight, and the reaction at right angles thereto, are correlated by an ordinary parallelogram of forces. Thus in Fig. 103, assuming the direction of flight to be horizontal, if W be the weight supported, R be the total normal reaction, and S be the force of propulsion, the relative magnitude of these forces will be given by the resolution shown. Expressing W and S in terms of R and the angle β, we have:—$W = R \cos \beta$ and $S = R \sin \beta$.

The quantity S is in many cases very small and difficult to measure directly, so that it is usually, for small values of β, deduced from the normal reaction, values of which have been given in Figs. 93 and 96.

It is evident that if the form of aerofoil under investigation be other than *plane* no such simple relationship as the foregoing exists; the vertical and horizontal components require to be measured independently.

The propriety of neglecting tangential forces has sometimes

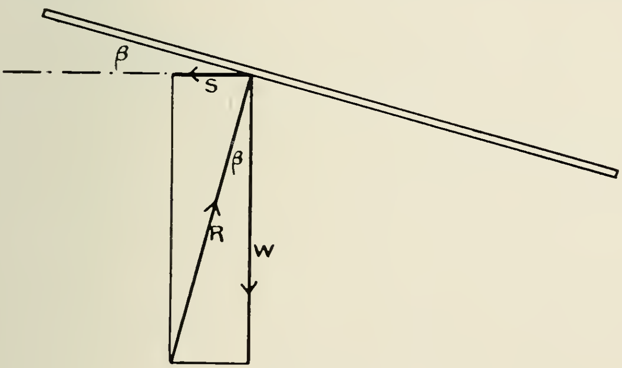

Fig. 103.

been questioned; such forces certainly cannot be neglected *on the grounds of their negligibility* (an error actually fallen into by Langley), and it is desirable to inquire into their exact nature in order that the consequences may be clearly understood and a correction provided.

If we had to deal with a plane devoid of thickness, so that its whole boundary surfaces might be said to lie in one plane, then there is only one possible kind of tangential force, *i.e.*, that due to the viscosity of the fluid; there must not only be viscosity within the fluid, but also physical continuity between the fluid and the plane itself capable of transmitting viscous stress. We could imagine this source of tangential force disposed of, either by making the fluid inviscid or by supposing the surfaces of the plane frictionless and not attached to the fluid in any way. Whichever be the assumption, the quantity that is being ignored is that known as "*skin-friction*," the general principles relating to which have been discussed in Chap. II.

In actual planes it is impossible to do away with thickness, so that in addition to skin friction there must be the possibility of a longitudinal pressure component due to the *shape* of the plane. Thus, if the plane be of "fair" form, *i.e.*, a stream-line solid based on an axis plane (Fig. 104), the pressure distribution, not being in any sense symmetrically disposed, may conceivably possess a longitudinal component of quite considerable value; or if the plane be of uniform thickness and square edges, as in the planes of Langley, we have no means of computing the edge

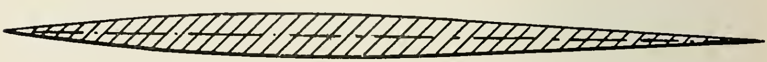

Fig. 104.

pressure resultant, for it is by no means certain that it can be represented by the resistance of the edge equivalent divested of its associations. There might, for example, in the types of motion illustrated in Fig. 98, be a region of negative pressure or *suction* on the front *edge* of the plane such as would entirely invalidate any ordinary computation.

§ 157. **The Coefficient of Skin-Friction.**—The hypothetical case of an aeroplane of zero thickness in edgewise motion offers the simplest possible case of skin-friction. The magnitude of the resistance due to this cause has been variously estimated, but at present is not known with any great degree of certainty. The value of skin-friction can be conveniently expressed as a coefficient, this coefficient being the resistance of a plane moving edgewise *in terms of* the resistance of the same plane when normal to the direction of motion. Reasoning from the facts known in connection with skin-friction in the case of water, we may infer that this form of resistance will vary approximately as the square of the velocity, but more accurately, proportionately to some power of the velocity rather less than the square, the index being lower than in the case of the normal plane. A consequence of this is that the "coefficient" will be greater for small planes at low velocities and less for larger planes at higher velocities.

Langley in his Memoir ("*Experiments in Aerodynamics*," pp. 9 and 25), and Hiram Maxim (*Century Mag.*, xlii., 829 and 836, 1891) have both stated explicitly that the influence of skin-friction in its relation to flight is negligible. Langley gives this result as a deduction from certain of his experiments, also as a matter of calculation based on Clerk Maxwell's value of the viscosity of air. He concludes from the latter that the frictional resistance is "less than 1/50 of one per cent. of that of the same plane moving normally," that is to say, he arrives at a coefficient of skin-friction of less than ·0002.

The author finds that Langley's deduction in this matter is not justified by the experiments upon which it is founded, and, further, that his calculation is based upon inadequate data and is in error.[1] The author has further shown, in Chap. VII., that skin-friction is *a dominating* factor in the economics of flight.

The direct measurement of skin-friction is a matter of considerable difficulty, so much so that experiments specially devised merely to detect its presence (as in the disc experiment of Dines)[2] have proved abortive. The author, by means of experiments (described in a subsequent chapter), has succeeded in measuring approximately the value of the coefficient of skin-friction ξ the following conclusions may be stated:—

(1) For smooth planes of a few square inches area at low velocities (about 10 feet per second), $\xi = {\cdot}02$ to ·025.

[1] Compare Chap. X.

[2] Dines, "On Wind Pressure upon an Inclined Surface" (*Proc. Royal Soc.*, *XLVIII.*, p. 243.

(2) For larger planes, ·5 to 1·5 square feet area, at higher velocities (about 20 to 30 feet per second), $\xi = \cdot 009$ to ·015.

(3) A plane of about $\frac{1}{2}$ square foot area, coated with No. $2\frac{1}{2}$ (Oakey's) glass paper gave, $\xi = \cdot 02$ (approx.).

(4) For *single surfaces* (as the surface of a stream-line body) the *half value* of ξ must be employed.

The experiments upon which the above results are based were made with planes of from 3 : 1 to 4 : 1 ratio in pterygoid aspect; the values are probably lower for square planes or planes in apteroid aspect. These experiments are still in progress.

§ 158. Edge Resistance in its Relation to Skin-Friction.—There is a subtle interaction between direct edge resistance and skin friction which merits discussion. Where the plane is bounded by square cut edges, or edges of bluff form, a certain amount of direct resistance is experienced. The work done from this cause is largely employed in setting in motion the air that impinges on the leading edge of the plane, and which afterwards " washes " its two surfaces. This has for a consequence the lessening of the skin- or surface- friction, for the air in contact with the plane, having already a velocity imparted to it, does not exercise so great a viscous drag. The influence of this edge effect is comparable to the diminution of the coefficient, as the distance from the " cut-water " is increased (discovered by Froude in the case of water); here the fluid, having been set in motion by the first part of the plane, does not exercise so great a drag on the part that follows. In a plane such as we are considering the total resistance will not be the sum of the edge resistance and skin-friction separately assessed, but will be less than this amount, and may be very little greater than the one or the other of the resistances measured separately.

It is probable that for *planes of less than a certain proportionate thickness* the augmentation due to the edge area is imperceptible, and that for such thin planes edge effect can be ignored. Equally it is probable that for planes of rectangular section of more than a certain proportionate thickness the skin-friction disappears and the total resistance may be assessed as edge effect.

Amongst the former may be classified laminæ of mica (such as used by the author) of an inch or a few inches breadth and 1/1000 inch or 3/1000 inch thickness. To the latter might be said to belong a plane of the proportions of a common floor-board. Probably the planes employed by Langley, about one square foot area, and of various proportions, by 1/10 inch thickness, would be intermediate, where edge effect and skin-friction give a total greater than either, but far less than their separately computed sum.

§ 159. Planes at Small Angles.—It commonly happens in physical problems that the conditions are greatly simplified when limited to the case of some particular angle being *small*, that is to say, within the range for which the angle (in circular measure), its sine, and its tangent, are sensibly equal to one another. In a case such as the present, where a high degree of accuracy is not important, and not attainable, such a range may be said to extend to as much as ten or fifteen degrees, and thus include practically the whole range of angle that can be usefully employed in the application of the aeroplane to aviation. It is consequently of importance to examine the extent to which simplification is possible under these restricted conditions.

It has been shown in the case of the square plane that Duchemin's formula: $P_\beta = P_{90} \dfrac{2 \sin \beta}{1 + \sin^2 \beta}$ does not greatly differ from the results of direct experiment, and we know that for small values of β the quantity $\sin^2 \beta$ may be neglected, so that the expression becomes: $P_\beta = P_{90} \times 2 \sin \beta$, or neglecting the difference between β and $\sin \beta$ (which for 10 degrees is less than 2 per cent.), we have: $P_\beta = P_{90} \times 2 \beta$, where β is expressed in circular measure.

The same form of expression is found to apply to planes generally for small angles, though the departure from the law with increase of angle would appear to be less in the case of the square plane than for planes of elongate form, whether in pterygoid or apteroid aspect. This is rendered evident by reference to Figs. 95, 96, 99. If the above form of expression held good each curve would be represented by a straight line passing through the origin. The actual curves are, in the vicinity of the

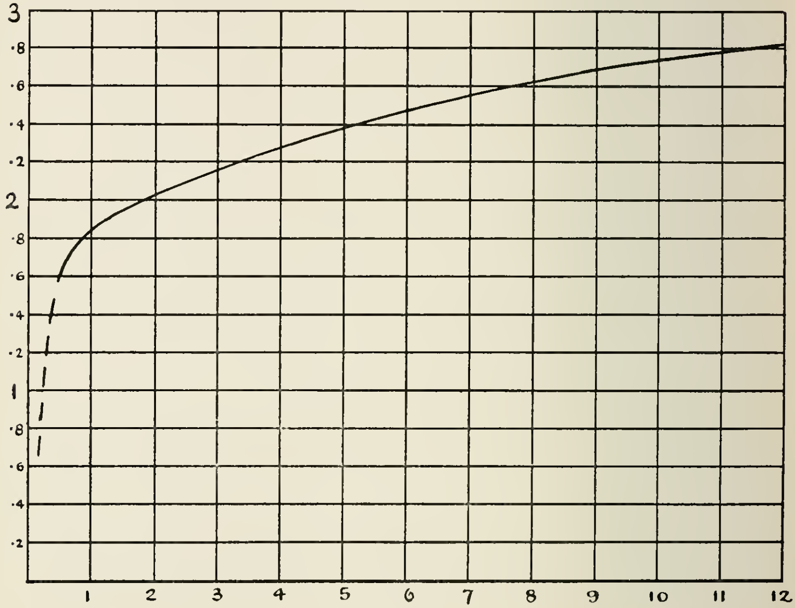

Fig. 105.

origin, sensibly straight, but the departure from the straight is more marked in the case of planes of extreme proportion, and the applicability of the straight line law in this case is consequently more restricted, or subject to greater error.

Let us write the general expression for small angles in the form $P_\beta/P_{90} = c\,\beta$, where c is a constant. Then the value of c which determines the slope of the line when plotted, depends upon the shape and aspect of the plane, or, in the case of a rectangular plane, its aspect and aspect ratio.

Fig. 105 illustrates the manner in which c may be plotted as a function of the aspect ratio n. The values of c are at present not known with any pretence to accuracy; c is probably different in the case of an aeroplane from what it is in the case of a pterygoid aerofoil.[1] For the former Langley found that variations in n gave rise to very considerable variations in c (Fig. 99); Dines failed to discover any variation at all (Fig. 101).

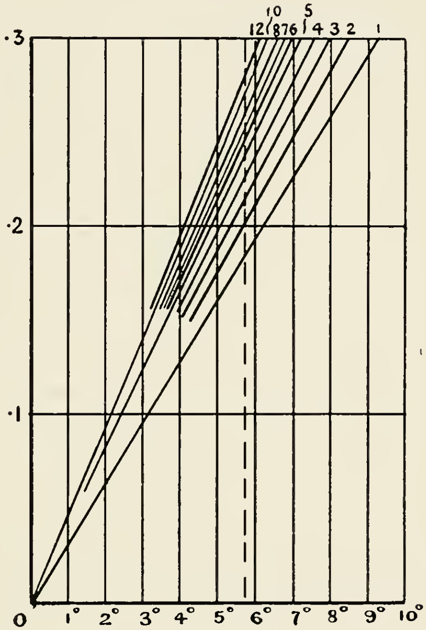

Fig. 106.

The values given in Fig. 105 are "plausible values" (see Chap. VIII.) for a pterygoid aerofoil. The same data have been laid out in Fig. 106, where abscissæ give angle and ordinates pressure reaction.

In addition to the equation,

$$\frac{P_\beta}{P_{90}} = c\,\beta \qquad (1)$$

we may also formulate as a direct consequence of the small angle hypothesis,

$$\frac{W}{A} = P_\beta \qquad (2)$$

and from the resolution of forces we have,

$$y = \beta\, W \qquad (3)$$

where y = aerodynamic resistance.

Consequently,

$$\frac{W}{A} = c\,\beta\, P_{90} = c\, C\, \rho\, \beta\, V^2$$

[1] See foot-note, § 172.

or, $$W \propto \beta A V^2 \qquad \beta \propto \frac{W}{A V^2} \tag{4}$$

and by (3) and (4),

$$y = \beta W = \frac{W^2}{c C \rho A V^2}$$

or, $$y \propto \frac{W^2}{A V^2} \tag{5}$$

or for given value of W,

$$y \propto \frac{1}{A V^2} \tag{5a}$$

Work done *aerodynamically* per second $= y V$, or $\propto \frac{W^2}{A V}$, or for given values of W and A, *Power* (*h.p.*) $\propto \frac{1}{V}$. (6)

We may interpret and summarise the above as follows:—

(1) The normal reaction of any given plane is proportional to its angle. The constant connecting the quantities depends upon the aspect ratio, and increases with the aspect ratio according to a law not at present known.

(2) The weight supported is sensibly equal to the normal reaction.

(3) Neglecting skin-friction and edge effect, the resistance in the line of flight varies as the angle multiplied by the weight sustained.

(4) Other things being equal, the weight supported varies as the square of the velocity.

(5) Neglecting skin-friction and edge effect, the resistance in the line of flight is directly as the square of the weight sustained and inversely as the area and the square of the velocity.

(6) Neglecting skin-friction and edge effect, the work done per unit time, *i.e.*, the power required for a given weight sustained and a given area, varies *inversely as the velocity of flight.*

§ 160. The Newtonian Theory Modified; the Hypothesis of Constant "Sweep."—In the theory of the Newtonian medium, for a given velocity the mass of fluid dealt with is proportional to the sine of the angle β; in a real fluid it is evident that the particles cannot cross each other's paths as depicted in Fig. 92, but will be constrained to move in a congruent manner. Thus if one layer be supposed to strike the plane and follow its surface, the next layer will be in turn deflected and move parallel to the first, and so on. If the particles of fluid were artificially constrained so as to be unable to undergo any change of velocity along the axis of flight, or to spread laterally, this influence would be transmitted from layer to layer with undiminished amplitude, or in the case of an elastic fluid until the initial displacement had been absorbed by compression. If we suppose the artificial constraint to be removed, then the amplitude rapidly diminishes as we get further

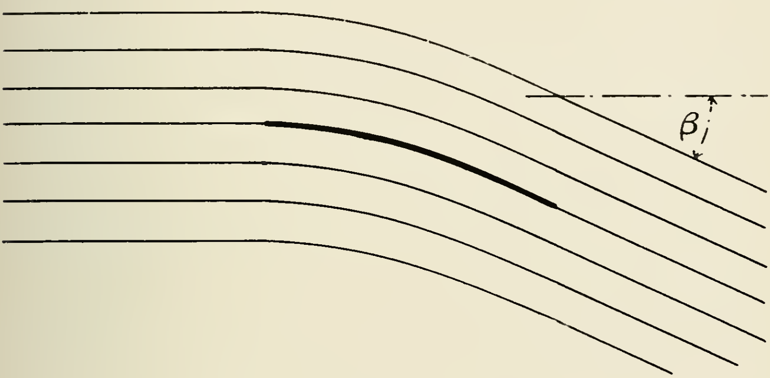

Fig. 107.

from the plane owing to the longitudinal motions of the fluid particles ; this may be regarded as a leakage of the fluid round the plane from the compression to the rarefaction side. (Compare Chap. IV., § 109.)

Now the facility with which the air or fluid can escape round the plane from one side to the other is evidently, for small angles at any rate, independent of the angle and dependent only on the size and shape of the plane, and for planes of elongate form it evidently depends largely upon the smaller dimension and to a less extent upon the greater. Thus in the case of a plane in pterygoid aspect the thickness of the layer affected by the passage of the plane will depend upon the dimensions of the latter and not upon its angle, and for a given plane the thickness of the layer will be constant.

In the foregoing paragraph the term "thickness" is used somewhat loosely. It is evident that there is no definite point at which the influence ceases altogether; and this brings us to a convention which it is found advantageous to adopt.

Let us suppose that the plane be supported by a definite stratum of air to which a uniform downward motion is imparted (Fig. 107)[1]; let us term the vertical cross-section of this stream or stratum the "*sweep*" of the plane and denote its downward velocity by v.

Then it is clear that for similar planes the *sweep* will bear a definite constant relation to the area A; let us, as in § 109, denote the *sweep* by the symbol κA where κ is a constant proper to the shape of the plane; in the case of rectangular planes a given value of κ will correspond to some definite aspect ratio.

Now the mass of air handled per second will be $= \rho \kappa A V$, and the momentum $= \rho \kappa A V v = \rho \kappa A V^2 \sin \beta$, which for small angles $= \rho \kappa A V^2 \beta$, where β is in circular measure. We therefore have $P_\beta = \rho \kappa V^2 \beta$, that is, $\frac{P_\beta}{P_{90}} = \frac{\rho \kappa V^2 \beta}{C \rho V^2} = \frac{\kappa}{C} \beta$ under the conditions of the present hypothesis.

But by § 159 we know from experiment that for small angles (such as under discussion) $\frac{P_\beta}{P_{90}} = c \beta$ where c is a constant depending upon the plane form and aspect; thus our hypothesis leads us to an expression of the correct form.

If we endeavour to deduce the constant κ from c and C (constants experimentally determined and known) from the resulting equation, $\kappa = c C$, we obtain a value far in excess of

[1] The curved section shown in Fig. 107 relates to the subsequent discussion (Chap. VIII.); it is perhaps easier to represent the conception on which the hypothesis of *constant sweep* is based by showing a curved section than a plane; in the latter case the flow has to be shown of angular path. The difference is otherwise unimportant.

that indicated by the experiment of the superposed plane (§ 154), hence it is evident that the hypothesis is insufficient.

§ 161. Extension of Hypothesis.—According to the principles laid down in Chaps. III. and IV., the neighbourhood of a plane or other aerofoil sustaining a load becomes the seat of a cyclic disturbance, and the air in advance of the aerofoil is in a state of upward motion; it has been shown that this up-current contributes to the supporting power of the plane or aerofoil, that is to say, its momentum contributes to the total load carried.

Let us represent this cyclic disturbance by supposing that in Fig. 107 the air stratum, instead of meeting the plane horizontally, has an upward component so that its motion (plotted relatively to the plane) be inclined at an angle α (Fig. 108), so that its upward velocity will be $V \sin \alpha$, or for small angles $V \alpha$.

Then the mass per second will be $\rho\ \kappa\ A\ V$ as before, and the momentum $= \rho\ \kappa\ A\ V^2\ (\alpha + \beta)$ or $\frac{P_\beta}{P_{90}} = \frac{\kappa}{C}\ (\alpha + \beta)$.

But we know by § 159 that $\frac{P_\beta}{P_{90}} = c\ \beta$, so that we now have the equation—

$$\frac{\kappa}{C}(\alpha + \beta) = c\ \beta, \text{ whence } \frac{\alpha + \beta}{\beta} = \frac{c\ C}{\kappa}$$

or

$$\frac{\alpha}{\beta} = \frac{c\ C}{\kappa} - 1.$$

Thus for any given plane, C and c being known experimentally, and κ being estimated from trials of superposed planes, we can calculate the equivalent up-current due to the cyclic disturbance, *within the limits of the present hypothesis*. This qualifying phrase is necessary because the supposed motion of the fluid, as depicted in Fig. 108, is conventional, and it is only on this conventional basis that we have effected a solution. The theory on the present lines is more fully developed in Chap. VIII., where it is made to perform useful work. The author, however, does not regard it as by any means final; the theory of the future should be based on a more comprehensive treatment of the whole motion of the fluid, in which the pressure reaction should appear as an integration; the present theory may be said to be based on the assumption that this integration of the whole motion of the fluid may be fairly represented on the hypothesis of a finite layer uniformly acted upon.

While pointing out the imperfect nature of the hypothesis at present adopted, it is perhaps fair to say that its defects are comparable to those of the Rankine-Froude method of dealing with the problem of propulsion, and in common with that

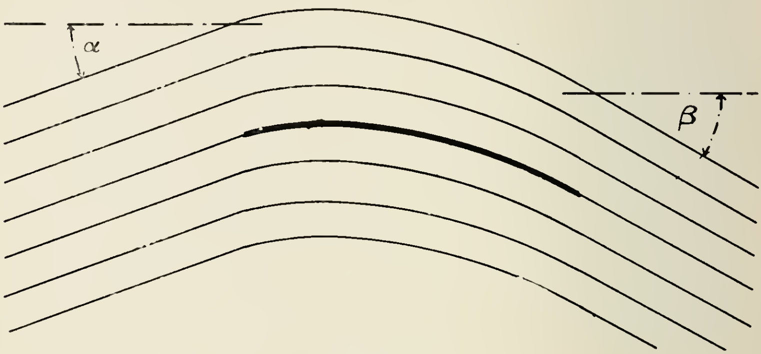

Fig. 108.

method it may be found to perform all that is practically required.

At present there are some difficulties, as will appear when the method is more fully discussed; these difficulties relate principally to the application of the somewhat unreliable data at present available—in particular, estimates of the value of κ from existing data can be little more than guess-work, and it is questionable whether experiments conducted with merely a pair of planes are sufficient; in all probability the true value can only be obtained when a veritable *screen of planes* is employed.

It would appear highly probable that a separation that might be sufficient to prevent loss of pressure where two planes only are superposed would prove quite insufficient if a greater number of planes were involved, for, according to § 122 (Fig. 73), the individual systems of flow fuse into one greater system, and are not, as Langley supposed, independent, consequently they will each and all react on one another, and the more numerous they become the wider they will require to be separated.

A limiting width will evidently be approached asymptotically when the number of planes becomes very great, and the limiting condition is that which most nearly resembles that of our hypothesis, for the whole depth of the fluid is then acted on with approximate uniformity, and the sweep of each plane will be fairly represented by the area included between any two adjacent planes. *Hence the value of κ deduced from experiments with pairs*

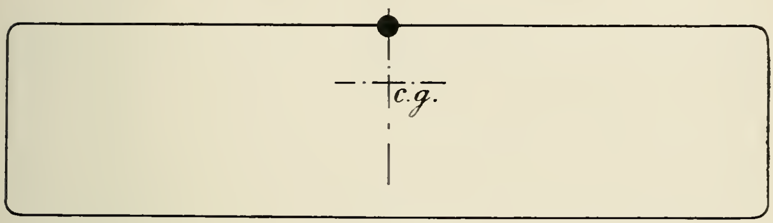

FIG. 109.

of superposed planes may be less than its true value according to the requirements of hypothesis.

§ 162. **The Ballasted Aeroplane.**—It has long been known that an aeroplane suitably ballasted will exhibit a certain degree of stability, and may be regarded in fact as a rudimentary *aerodrome.* This fact is mentioned by Moulliard in his "Empire de l'Air" (1881), who, however, bases his discussion on a quite fantastic theory involving a supposed *change in the position of the centre of gravity due to changes of velocity.*[1]

[1] That so keen an observer as M. Moulliard should have fallen into so extraordinary an error is almost incredible; the following passage, however, occurs in his work: "Avant d'aller plus loin, je suis forcé d'énoncer une propriété de l'attraction sur les corps en mouvement: propriété qui est connue ou inconnue, je ne sais; mais qui en tout cas existe, c'est celle-ci: *Quand un corps se meut, son centre de gravité se deplace, et se transporte en*

Let an aeroplane (Fig. 109) be loaded in the manner shown so as to bring its centre of gravity to a position from a quarter to one-third of its width from one of its edges (we may assume it to be a plane of about 4 : 1 ratio in pterygoid aspect), and let it be launched in free flight with the ballasted edge leading to its line of flight. It will then be found that, provided the air is sufficiently calm, the plane will glide after the manner of a bird in passive flight, and will show itself to be possessed of complete stability.

The ballasted aeroplane in free flight may be employed for the determination of aerodynamic data as follows[1] :—

(1) The value of c for planes of different aspect ratio in the expression, $P_\beta = c\, \beta\, P_{90}$.

(2) The determination and plotting of the position of the centre of pressure as a function of the angle of inclination for small angles.

(3) The determination of the value of the coefficient of skin-friction, ξ.

The most satisfactory results can be obtained by employing planes of mica, of only a few thousandths of an inch in thickness, the ballasting being effected by a split lead shot, as shown in the figure. Such planes show a perfection of equilibrium that appears to be unattainable with any other material than mica; it is also important that the ballast should be applied in a compact mass centrally and not distributed along the front edge. In order to improve the "sense of direction" it is found to be advantageous to "dog-ear" the front corners, slightly turning them *upwards*. It may be further noted that the rectangular form is very advantageous; in general other forms give inferior stability.

The theory of the equilibrium of the ballasted aeroplane belongs more correctly to the domain of *aerodonetics*, but the

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arrière du sens du mouvement." These words leave no loophole for a second interpretation, and even if they did so, the subsequent argument leaves no vestige of doubt.

[1] See account of author's experiments, Chap. X.

importance of the matter warrants its premature introduction as touching the aerodynamic aspect of the subject.

Let us suppose that the position of the centre of gravity be such as will coincide with the centre of pressure when the plane makes an angle $= \beta_1$ with its direction of motion. Now we know (§ 184) that the position of the centre of pressure varies as a function of β and that its distance from the front edge of the plane diminishes the less the angle; if then the angle from any accidental cause becomes less than β_1 the centre of pressure will move forward in advance of the centre of gravity so that the forces acting on the plane will form a couple tending to increase the angle and so restore the condition of equilibrium. Likewise if the angle become too great the centre of pressure will recede and the resulting couple will tend to diminish the angle, and again the equilibrium is restored; thus the conditions are those of stable equilibrium, the plane tends to maintain its proper inclination to its line of flight.

There is not only equilibrium between the *angle of the plane* and its *direction of motion* as above demonstrated, but also between the *gliding angle* and the *velocity of flight;* thus if the velocity is deficient, so that the weight is insufficiently sustained, the gliding angle and the component of gravity in the line of flight automatically increase and the aerodrome undergoes acceleration. Conversely, if the velocity is excessive, the gliding angle (and so the propulsive component) diminishes, and the velocity is thereby reduced.[1]

[1] The above explanation of the automatic stability of an aerodrome is, in a condensed form, that given by the author in his paper to the Birmingham Natural History and Philosophical Society in 1894.

CHAPTER VII.

THE ECONOMICS OF FLIGHT.

§ 163. **Energy Expended in Flight.**—There are certain general propositions relating to the Economics of Flight that may now be demonstrated, and which are essential to the further development of our subject.

The energy expended in flight is utilised in two directions: firstly, in the renewal of the aerodynamic disturbance, or wave necessary to the support of the weight, that is, the energy *expended aerodynamically;* secondly, the energy expended in overcoming the direct resistance, *i.e.*, that due to skin friction and eddy making, which varies approximately as the square of the velocity. Let the latter be denoted by the symbol x, and let y be the aerodynamic resistance.

Now we have seen that the aerodynamic resistance varies approximately in the inverse ratio of the velocity squared (§§ 159 and 160), for any given weight sustained, so that if we take the case of an aerodrome supporting a given load (inclusive of its own weight) we have the relation, $y \propto \frac{1}{V^2}$, and if we further assume the factors which give rise to direct resistance to undergo no change, we have, $x \propto V^2$. And the total resistance $= x + y$, $\therefore$ the energy expended in flight *per unit distance* $= x + y$, and energy *per unit time,* or *power* $=$ $V \times (x + y)$.

§ 164. **Minimum Energy. Two Propositions.** — Taking the achievement of flight for granted, the problem of least energy presents itself in two forms :

(1) To determine the conditions under which the greatest distance may be covered on a given supply of energy, that is to say, the conditions of *least resistance ;*

(2) To remain in the air for the longest possible time on a given supply of energy, that is, to determine the conditions of *least horse-power.*

Prop. I.—We have—

By § 157, $x \propto V^2$ and by § 159 (5), $y \propto \frac{1}{V^2}$ or $x \propto \frac{1}{y}$

$$\therefore \qquad \frac{dx}{dy} = -\frac{x}{y} \qquad (1)$$

Now conditions are fulfilled when $x + y$ is minimum, that is when $dx = -dy$, or $\frac{dx}{dy} = -1$, $\therefore$ by (1), $-\frac{x}{y} = -1$,

or, $$x = y$$

Therefore, under the conditions of hypothesis, *an aerodrome will travel the greatest distance on a given supply of energy when its aerodynamic and direct resistances are equal to one another.*

Prop. II.—We have—

xV power expended (energy per second) in overcoming direct resistance.

yV power expended (energy per second) in overcoming aerodynamic resistance.

Then $xV \propto V^3$ and $yV \propto \frac{1}{V}$

Denote xV by X and yV by Y, we have $X \propto \frac{1}{Y^3}$ or,

$\frac{dX}{dY} = -3\frac{X}{Y}$, and when $dX = -dY$ we have $Y = 3X$

or, $$y = 3x.$$

Therefore, *an aerodrome will remain in the air for the longest possible time on a given supply of energy, that is to say, its flight will be accomplished on least horse-power, when the resistance due to aerodynamic support is three times the direct resistance.*

On the foregoing propositions a third may be founded as follows :—

Prop. III.—To determine the relation of the speed of greatest range to the speed of least power.

Now $x = kV^2$ and $y = n\frac{1}{V^2}$, where k and n are constants.

When $x = y$ let $V = V_1$.

When $3x = y$ let $V = V_2$. It is required to find the relation of V_1 to V_2.

When $x = y$ we have $k V_1^2 = \frac{n}{V_1^2}$ or $k = \frac{n}{V_1^4}$.

When $3x = y$ we have $3 k V_2^2 = \frac{n}{V_2^2}$.

Substituting for k we have—

$$3 \frac{n}{V_1^4} V_2^2 = \frac{n}{V_2^2}, \text{ or } 3 = \frac{V_1^4}{V_2^4}, \text{ or } \frac{V_1}{V_2} = 3^{\frac{1}{4}} = 1{\cdot}315$$

That is to say, *the speed of greatest range is* 1·315 *times the speed of least power.*

Corollary to Prop. III.—For a plane aerofoil the change in value of the angle β involved in the change of velocity from V_1 to V_2 can be immediately deduced.

$V_1 = 3^{\frac{1}{4}} V_2$, but by § 159, $V^2 \beta \propto W$ (for small angles). Consequently $V_1^2 \beta_1 = V_2^2 \beta_2$ where β_1 and β_2 are the angles appropriate to the velocities V_1 and V_2 respectively. Therefore—

$$\beta_1 (3^{\frac{1}{4}})^2 V_2^2 = \beta_2 V_2^2$$

or,

$$\beta_2 = \sqrt{3} \times \beta_1.$$

Thus calculations of β values for least resistance require to be multiplied by $\sqrt{3}$ to give appropriate values for least horse-power. We may thus anticipate that birds whose object in flight is to *fall as slowly as possible* (as birds whose habit is to be sustained on an upcurrent, and so to take advantage of the least upward velocity possible), will have wings of hollower form than those whose object is to get from point to point.

§ 165. Examination of Hypothesis.—According to the hypothesis on which the foregoing propositions are founded, it is supposed that a constant weight is sustained at a varying velocity by an aerofoil of constant area, so that on the one hand the resistance due to skin-friction for any stated velocity undergoes no change, and on the other that the law $y \propto \frac{1}{V^2}$ shall be applicable, this law being that ascertained as pertaining to an aeroplane for small angles, and deduced generally in § 160 from the hypothesis of constant "sweep."

So long as the foregoing hypothesis applies it is not important whether the direct resistance is entirely due to the skin friction of the aerofoil or whether it is in part due to the resistance of the "body" of the aerodrome, *i.e.*, that part that may be supposed to constitute or contain the load. If we require to concern ourselves with changes of aerofoil or "sail" area, it becomes necessary to distinguish between these two kinds of resistance, the total resistance x being supposed to be divided into two parts, the one x_1 being defined as independent of the sail area and the other x_2 as dependent and as directly proportional thereto. In all cases the approximate assumption is made that this class of resistance is proportional to velocity *squared*, the error that may result from this assumption being considered later.

Prop. IV.—*To determine the conditions controlling the aerofoil area for an aerodrome of given weight travelling at a specified velocity.*

Let A = area, then, since x_1 is fixed by the conditions, x_2, y, and A are the variables with which we are concerned :—

$$x_2 \propto A \quad \text{and} \quad y \propto \frac{1}{A} \qquad \text{(By § 159)}$$

or,

$$x_2 \propto \frac{1}{y}$$

that is,

$$\frac{dx_2}{dy} = -\frac{x_2}{y}$$

and as in prop. i. we have the minimum condition fulfilled when $x_2 = y$,

Therefore *the correct area has been given to the aerofoil when its aerodynamic resistance is equal to its direct resistance.*

Thus, if for the given conditions of weight and velocity the aerofoil be made too large, the skin friction will be in excess of the aerodynamic resistance; if insufficient surface is provided, the aerodynamic resistance will be in excess; in neither case will the energy required for a given distance be the least possible.

It may be noticed that this result is *in appearance* out of harmony with prop. i., for there it was shown that the most favourable velocity at which to run an aerodrome in order to cover the greatest distance on a given quantity of energy is that at which the aerodynamic resistance is equal to the *total* direct resistance, that is x; whereas according to the present proposition the most economical conditions are met with when $y = x_2$, which is only a portion of the total.

The explanation of this apparent paradox will be given in the light of the subsequent proposition.

§ 166. Velocity and Area both Variable.—Prop. V.—Given that $x_1 = 0$, then, for an aerodrome of given weight, with V and A both variable, *find the velocity at which a given flight (distance) can be accomplished with least energy.*

By § 159 (5a) $$y \propto \frac{1}{AV^2}$$

and, $$x_2 \propto AV^2$$

Now for $x_2 + y$ minimum, we have $x_2 = y$ (by Prop. IV.), or,—

$$AV^2 \propto \frac{1}{AV^2}$$

$$\therefore \quad (AV^2)^2 = \text{constant}.$$

hence, $$y = \text{constant}.$$

and, $$x_2 = \text{constant}.$$

$$\therefore \quad x_2 + y \text{ is constant},$$

or *the resistance is independent of the velocity.*

That is to say, if for an aerodrome of given weight the velocity be supposed to undergo continuous variation, and the "sail area" also undergo corresponding variation, so that the latter is at every moment so proportioned to the former as to result in the least possible resistance (in accordance with prop. iv.), then the total resistance of the aerodrome will be constant in respect of velocity and the energy required to pass from any one point to any other point will be constant, no matter what the speed may be.

Cor. I.—If the body resistance (x_1) be taken into account, the total resistance may be taken as composed of two parts, the one part which includes the x_2 and y of the equations and which is constant, and the other part x_1 which varies approximately as the square of the velocity, and results in making flight at high speeds, distance for distance, less economical than at low speeds.

Cor. II. — The conditions of greatest economy for a given aerodrome as enunciated in prop. i. will not be those of best value of area A, as laid down in prop. iv., unless the aerodrome have zero body resistance, for, the influence of body resistance being always to make low velocities more economical than high velocities, the velocity of least energy (per unit distance) will be less than that for which the aerodrome is correctly proportioned. This is the explanation of the apparent paradox of § 165. If we imagine an aerodrome designed for a given velocity, so that $x_2 = y$, then we could reduce its expenditure of energy, for given distance, by reducing its velocity till $x_1 + x_2 = y$ (that is, $x = y$, prop. i.), then by re-designing its area till once more $x_2 = y$ we can again render it more economical; this could be repeated *ad infinitum*, the economy increasing at each step, the net result, however, merely being the saving effected by transferring the "body" less rapidly through the air.

Cor. III.—The constancy of $x_2 + y$ demonstrated in the present proposition has for an immediate consequence the constancy of the *gliding angle* (if x_1 be ignored), that is to say, the thrust required to maintain an aerodrome in flight will be constant for a given value of W, and if this thrust be supplied by a component of gravity (Fig. 110), and if γ be the angle of descent, we have :— $\frac{Thrust}{Weight} = sin\ \gamma$, that is to say, *$\gamma$ is constant.* If we take account of the body resistance x_1, we shall find that the value of γ will increase the higher the velocity. This effect is more fully investigated in the subsequent section.

§ 167. The Gliding Angle, as affected by Body Resistance.—Let γ, as before, stand for the theoretical (constant) gliding angle when the body resistance is zero.

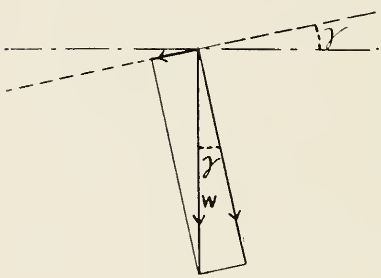


Fig. 110.

Let γ_1 be the gliding angle when the body resistance is x_1; then :—

When the total resistance is $x_2 + y$ gliding angle $= \gamma$, and when total resistance is $x_1 + x_2 + y$ gliding angle $= \gamma_1$, the weight W being the same in both cases.

Consequently we have for small angles $\frac{\gamma_1}{\gamma} = \frac{x_1 + x_2 + y}{x_2 + y}$, which for a correctly designed aerodrome, when $x_2 = y$, becomes—

$$\frac{\gamma_1}{\gamma} = \frac{x_1 + 2\,x_2}{2\,x_2}.$$

Taking as an illustration the case of a bird, and estimating the relation of x_1 to x_2 on the basis of skin-friction alone (which is probably near the truth), we find by measurement of different species that the body surface is at least $\frac{1}{3}$ of the wing surface, that is to say, we may take it that $x_2 = 3\,x_1$ and we have— $\frac{\gamma_1}{\gamma} = \frac{7}{6}$; that is to say, under the most favourable circumstances in bird flight the gliding angle is increased about $\frac{1}{6}$ by body resistance above what it would be were such resistance absent. In most game birds and other fast fliers the proportion would work out very much higher.

In the above illustration we have assumed that the bird has been correctly designed by Nature *on the basis of our present hypothesis.* There is an item of some importance which we have hitherto neglected and which will be subsequently taken into account, *i.e.*, the influence of *sail area on total weight.* We have so far assumed that the weight is constant and that the sail area may be increased or diminished at will, whereas in reality a part of the total weight is due to the wings themselves, and the total weight should be represented as some function of the area $(F)\ A$, *plus* a constant. This extension of the subject will be left for later investigation ; for the present we will continue to exhaust the problem from the present standpoint.

§ 168. Relation of Velocity of Design to Velocity of Least Energy. It has been pointed out in respect of props. i. and iv., in Cor. II., prop. v., that the *velocity for which an aerodrome is correctly designed* to cover the greatest distance *on a given supply of energy is* not the velocity *at which it will actually cover the greatest distance,* unless the body resistance is zero. Let us put the matter in the form of a further proposition :—

Prop. VI.—Given the relation of x_1 to x_2, determine the *velocity of least resistance* in terms of the velocity for which the aerodrome is *designed for least resistance.*

Let us represent the *designed velocity* by the symbol $\mathbf{V}$, and let V_1 (as in prop. iii.) represent the velocity of least resistance, that is, when $x = y$ (prop. i.). Then at velocity $\mathbf{V}$ we have $x_2 = y$, and at velocity V_1 we have $x_1 + x_2 = y$, where x and y are variables.

Let a_1 and a_2 represent normal areas that will give rise to resistances equal to x_1 and x_2.

Then, $x_1 = n\ a_1\ V^2$

$x_2 = n\ a_2\ V^2$ where n is a const.

and, $y = m \frac{1}{V^2}$ where m is a const.

When $V = \mathbf{V}$, $x_2 = y$,

$$\therefore \quad n\, a_2\, \mathbf{V}^2 = m\, \frac{1}{\mathbf{V}^2} \quad \text{or} \quad a_2\, \mathbf{V}^4 = \frac{m}{n} \tag{1}$$

When, $V = V_1, \qquad x_1 + x_2 = y,$

$$\therefore \quad n\,(a_1 + a_2)\, V_1^2 = m\, \frac{1}{V_1^2} \quad \text{or} \quad (a_1 + a_2)\, V_1^4 = \frac{m}{n}. \tag{2}$$

By (1) and (2) we have—

$$(a_1 + a_2)\, V_1^4 = a_2 \mathbf{V}^4$$

that is,

$$\frac{V_1^4}{\mathbf{V}^4} = \frac{a_2}{(a_1 + a_2)} = \frac{x_2}{x_1 + x_2}$$

or,

$$\frac{V_1}{\mathbf{V}} = \sqrt[4]{\frac{x_2}{x_1 + x_2}}.$$

The signification of this result is that if an aerodrome be designed to travel at a velocity $\mathbf{V}$, its " sail area " being such as will involve the least total resistance at that velocity, such an aerodrome will experience its least resistance when its velocity is reduced to—

$$\mathbf{V} \times \sqrt[4]{\frac{x_2}{x_1 + x_2}}.$$

As an example we may, as before, assign relative values $x_1 = 1$ $x_2 = 3$, we have velocity of least resistance,

$$V_1 = \sqrt[4]{\frac{3}{4}} \times \mathbf{V} = \cdot 93\ \mathbf{V}.$$

If we take $x_1 = x_2$ we shall have—

$$V_1 = \sqrt[4]{\frac{1}{2}} \times \mathbf{V} = \cdot 84\ \mathbf{V}.$$

Least Horse-power.—If we require to know the velocity of *least power* V_2 we have by prop. iii.: $V_2 = \frac{V_1}{3^{\frac{1}{4}}} = \cdot 76\ V_1$, or in terms of $\mathbf{V}$ we have—

$$V_2 = \mathbf{V} \times \sqrt[4]{\frac{x_2}{3\,(x_1 + x_2)}}.$$

In the case of the values given above,

When $x_2 = 3\, x_1 \qquad V_2 = \cdot 706\ \mathbf{V}.$

When $x_2 = x_1 \qquad V_2 = \cdot 638\ \mathbf{V}.$

§ 169. Influence of Viscosity.—The influence of viscosity in the resistance of bodies is to cause a departure from the V-square law. It has been shown (Chap. II.) that the resistance in a viscous fluid can be expressed as a power of the velocity whose index must be less than 2, this form of expression not representing a definite law that holds good over any wide range, but rather defining the rate of change in the quantities round about the values for which the index value may have been determined (§ 40).

Adopting this approximate form of expression, we shall have in prop. i. $x \propto V^n$, and assuming (as we are probably justified in assuming) that viscosity has only a negligible influence on the aerodynamic resistance, we have:—

$$x \propto V^n \quad y \propto \frac{1}{V^2}, \quad \therefore \quad x \propto \left(\frac{1}{y}\right)^{\frac{n}{2}}$$

∴ differentiating, we have—

$$\frac{dx}{dy} = -\frac{nx}{2y}.$$

Now $x + y$ is minimum when $dx = -dy$, that is—

$$-\frac{n\,x}{2\,y} = -1, \quad \text{or} \quad n\,x = 2\,y, \quad \text{or} \quad x = \frac{2}{n}\,y.$$

This is the solution to the equivalent of prop. i., on the modified hypothesis.

Thus if $n = 1{\cdot}75$, that is to say, if $x = V^{1{\cdot}75}$, we have the minimum total resistance when $x = \frac{8}{7}\,y$.

The necessary modifications to generalise the further propositions in respect of the index of x may be easily effected; the matter, however, has not been pursued further in the present work, the approximate assumption of $n = 2$ being deemed sufficient for the needs of the practical application of the theory.

§ 170. The Weight as a Function of the Sail Area.—It has been pointed out, in § 166, that we cannot, strictly speaking, regard the weight of an aerodrome as constant in respect of the value of A, for the supporting members themselves must possess weight, and such weight must be some function of the area. This consideration will not affect the results given by props. i., ii., iii., for in these the hypothesis does not contemplate any change in the value of the " sail area " A.

We may regard the total weight W of an aerodrome as consisting of two parts, W_1 and W_2, of which W_1 is constant being the weight of the essential load, and W_2 as the weight of the aerofoil which must vary in some way with the area A, or $W_2 = (F)\ A$ where the nature of the function must depend upon the conditions of design and construction.

Before any attempt can be made to investigate the influence of the matter under consideration, some assumption must be made as to the form of the function in question. The basis on which we have to found our assumption is that of some probable *constructive method;* thus we might suppose that as the aerofoil undergoes change of area its *geometrical form* is preserved and remains constant. If we take L as representing the linear dimension of the aerofoil (L may be chosen as any linear dimension so long as it is the same in all cases); then if the weight of the aerofoil per unit area were constant, we should have W_2 varies as L^2; or suppose we base our relationship on an assumption of constant geometrical form, but all *scantlings* of appropriate strength, investigation gives W_2 varies as L. The actual relation, whatever it may be, depends upon the exigencies of design and can be established for any set of conditions empirically by designing aerofoils of different area and plotting an $L : W_2$ curve.

In detail we find that the weight of each element of the aerofoil structure may be represented by the simple expression $k\ L^q$ where k and q are constants which are different for the different functional elements. Now we know that an expression consisting of the sum of a number of quantities of the form $k_1\ L^{q_1}$, $k_2\ L^{q_2}$, $k_3\ L^{q_3}$, etc., may be approximated over a moderate range by a simple expression of like form, and the approximation is greater the less the variation in the value of q in the component terms. Now in the present case in any reasonable design q is found to lie for every structural component between 1 and 2, so that we shall be justified in assuming the expression $W_2 = k\ L^q$ as approximately applicable.

§ 171. The Complete Equation of Least Resistance.—In prop. i. of the present chapter we investigated the conditions of least resistance in the simple case of an aerodrome of fixed weight and sail area. In prop. v. (cors. i. and ii.) we have dealt with the influence of a *body resistance* independent of the aerofoil area. In the present section it is proposed to generalise and include in the investigation the influence of the weight of the aerofoil as a variable, assuming the form of expression deduced in the preceding section.

It has been shown that the effect of body resistance is to make the resistance at high speeds greater than that at lower speeds; but we know that at low velocities the sail area requires to be increased and that consequently the weight becomes greater and the resistance will be increased on this account, and when the velocity becomes less than a certain value the increase of resistance from this cause will more than compensate for the decrease due to the reduction in the direct body resistance. We may therefore anticipate that the resistance has a minimum value at some definite velocity at which A, and consequently L, will have some definite ascertainable value.

Let W = total weight.

,, W_1 = constant *essential* weight.

,, W_2 = variable weight, dependent upon L.

,, A = aerofoil area.

,, L = a linear dimension which we may take to be $\sqrt{A}$.

,, y = aerodynamic resistance, of which—

y_1 is that due to W and

y_2 is that due to W_2.

Let x = direct resistance, of which—
x_1 is that due to body resistance, and
x_2 is that due to aerofoil area.
,, a_1 = a normal plane area whose resistance is the equivalent of x_1, and
,, a_2 = a normal plane area whose resistance is the equivalent of x_2.
,, V = velocity of flight.
,, ξ = coefficient of skin friction.

k and q are constants, as in preceding section, and $C_1\, C_2\, C_3$ are further constants.

It can be shown that—

$$y = C_1 \frac{W^2}{L^2\, V^2}, \quad \text{and} \quad x = C_2\, a_1\, V^2 + C_3\, \xi\, V^2\, L^2.$$

We require to know the minimum value of $x + y$; or, we require to solve for minimum value the expression—

$$C_1 \frac{W^2}{L^2\, V^2} + C_2\, a_1\, V^2 + C_3\, \xi\, V^2\, L^2.$$

Now $W = W_1 + W_2$, and $W_2 = k\, L^q$, so that expression becomes

$$C_1 \frac{(W_1 + k\, L_q)^2}{L^2\, V^2} + C_2\, a_1\, V^2 + C_3\, \xi\, V^2\, L^2$$

or

$$C_1\, W_1^2 \frac{1}{L^2\, V^2} + C_1\, W_1\, 2\, k\, \frac{1}{L^{2-q}\, V^2} + C_1\, k^2 \frac{1}{L^{2-2q}\, V^2}$$
$$+ C_2\, a_1\, V^2 + C_3\, \xi\, V^2\, L^2,$$

where L and V are variables.

Making a temporary substitution of constants in order to abbreviate, we have—

$$\frac{a}{L^2\, V^2} + \frac{b}{L^{2-q}\, V^2} + \frac{c}{L^{2-2q}\, V^2} + e\, V^2 + f\, V^2\, L^2.$$

Differentiating in respect of L and V gives simultaneous equations as follows:—

$$-\, 2\, \frac{a}{L^3\, V^2} - (2 - q) \frac{b}{L^{3-q}\, V^2} - (2 - 2\, q) \frac{c}{L^{3-2q}\, V^2}$$
$$+\, 2 f\, L\, V^2 = 0. \qquad (1)$$

$$-\frac{a}{L^2 V^3} - \frac{b}{L^{2-q} V^3} - \frac{c}{L^{2-2q} V^3} + e V + f V L^2 = 0. \quad (2)$$

By (1)
$$V^4 = \frac{a}{f L^4} + \frac{(2-q) b}{2 f L^{4-q}} + \frac{(1-q) c}{f L^{4-2q}}. \quad (3)$$

By (2)
$$V^4 = \frac{a}{(e + f L^2) L^2} + \frac{b}{(e + f L^2) L^{2-q}} + \frac{c}{(e + f L^2) L^{2-2q}}. \quad (4)$$

Or, eliminating V, we have—

$$\frac{a}{(e + f L^2) L^2} + \frac{b}{(e + f L^2) L^{2-q}} + \frac{c}{(e + f L^2) L^{2-2q}} = \frac{a}{f L^4} + \frac{(2-q) b}{2 f L^{4-q}} + \frac{(1-q) c}{f L^{4-2q}}.$$

Simplifying and substituting for a, b, c, etc., we obtain—

$$\frac{C_3 \xi}{C_2 a_1 + C_3 \xi L^2} = \frac{W_1 - (q-1) k L^q}{L^2 (W_1 + k L^q)}. \quad (5)$$

This is the solution in its most general form, and gives the condition of least resistance. All the quantities except L are known to the designer of the aerodrome; the value of L determined from the equation gives the value of V from either Equation (3) or (4); it also immediately defines the area. The form of Equation (5) is such that it can only be solved by plotting.

If the necessary data to any aerodrome are known we can thus ascertain the velocity of least resistance and prescribe the correct "sail area." It is not always, however, that the general solution of the problem is desired—in fact, more frequently than not the value of V is prescribed by considerations external to the aerodynamics of the subject, when the problem becomes to determine the area of least resistance corresponding to the stated value of V. In this case the differentiation in respect of L is all that is necessary, and we fall back on Equation (1), V being a constant.

The practical application of the present investigation and the employment of the foregoing equation are discussed in the subsequent chapter.

CHAPTER VIII.

THE AEROFOIL.[1]

§ 172. Introductory.—At some future period it may be found possible to rationalise the treatment of the theory concerned with the form of the aerofoil on a comprehensive basis, so that the sectional form at every point shall be correlated to the pressure reaction and the strength of the cyclic disturbance. At present we are compelled to take our stand on a simplified and somewhat conventional hypothesis.

In the case of the aeroplane, in respect of which a certain amount of experimental data is available, we can at once proceed to apply the fundamental propositions of the preceding chapter, to determine the angle of least resistance, thus:—

Let, as before (§ 163), $x + y$ be the total resistance in the line of flight, where x is the direct resistance (due to skin friction, etc.) and y that due to work expended dynamically.

Then the condition of least resistance is that $x = y$.

Now $x = \xi\, AP_{90}$, and $y = \beta W = c\,\beta^2\, AP_{90}$ (for small values of β), or $\xi = c\,\beta^2$, that is, $\beta = \sqrt{\dfrac{\xi}{c}}$ where β is the angle of inclination in radians; ξ is the coefficient of skin friction (§ 157), and c is the constant according to § 159.

If β° be the value of the angle expressed in degrees, the expression becomes $\beta_c = \dfrac{180}{\pi} \sqrt{\dfrac{\xi}{c}}$.

Taking for example the case of a square plane for which the value of c is 2, and taking $\xi = {\cdot}02$, we have—

$$\beta^\circ = \frac{180}{3{\cdot}14} \times \sqrt{\frac{{\cdot}02}{2}} = 5{\cdot}7^\circ \text{ approximately.}$$

[1] See footnote, § 128.

A plane of elongate form in pterygoid aspect whose value of c is $= 3$ would thus have an angle of least resistance of slightly over $4\frac{3}{4}°$. This is about the minimum value that would in the ordinary way be obtained, assuming that correct values[1] have been assigned to c and ξ.

When we have to deal with an aerofoil of curvilinear section adapted to the form of the lines of flow, we may obtain useful results by adopting the hypothesis of *constant sweep* (§ 160). According to this hypothesis it is assumed that the support is derived from a layer or stratum of fluid uniformly acted on by the aerofoil, and whose cross-sectional area is constant. This area, for a given plan-form of aerofoil in stated aspect, is equal to the aerofoil area A multiplied by the constant κ, or, as given in § 160, we have, *sweep* $= \kappa A$.

It will be further assumed that the relation $\frac{a}{\beta}$ (§ 161) is constant for any given plan-form and aspect.

§ 173. The Pterygoid Aerofoil. Best Value of β—.

Let $\epsilon = \frac{a}{\beta}$ and, as before,

„ A = aerofoil area,

„ κA = sweep,

„ ξ = coefficient of skin friction.

C is the constant of the normal plane (§ 136).

Now the direct resistance $x = \xi \, A \, C \, \rho \, V^2$, and the aerodynamic resistance y is equal to the *energy expended aerodynamically per second divided by the velocity*, or $y = \frac{1}{2} \rho \kappa A V \times V^2 (\beta^2 - a^2) \div V = \frac{1}{2} \rho \kappa A V^2 (\beta^2 - a^2)$.

[1] The values of c for the aeroplane are probably not the same as for a pterygoid aerofoil of the same aspect ratio. Neither value has yet been determined with any degree of accuracy; the values given in Figs. 105 and 106 and in tabular form in § 177 are probably more nearly correct for the pterygoid form.

It will be shown subsequently that the effective value of ξ in the case of an inclined plane may be less than its true value.

But condition of least total energy is that $x = y$. Let $\beta = \beta_1$ and $a = a_1$ when $x = y$.

$$\xi\, C = \tfrac{1}{2}\, \kappa\, (\beta_1^2 - a_1^2) \qquad (1)$$

Now, $$\frac{a}{\beta} = \epsilon, \text{ or } a_1^2 = \beta_1^2\, \epsilon^2$$

or $$(\beta_1^2 - a_1^2) = \beta_1^2\, (1 - \epsilon^2)$$

(1) becomes $$\beta_1^2\, (1 - \epsilon^2) = \frac{2\, \xi\, C}{\kappa}$$

or $$\beta_1 = \sqrt{\frac{2\, \xi\, C}{\kappa\, (1 - \epsilon^2)}}. \qquad (2)$$

If β is expressed in degrees this becomes—

$$\beta_1{}^\circ = \frac{180}{\pi} \sqrt{\frac{2\, \xi\, C}{\kappa\, (1 - \epsilon^2)}}. \qquad (2a)$$

Of the quantities involved in this expression, ξ and C are known by experiment; κ also may be experimentally determined by the method of superposed planes discussed in §§ 154 and 161; the experimental data are, however, at present wanting. For planes and other forms of aerofoil in pterygoid aspect κ is a function of the aspect ratio and is greater when the aspect ratio is greater. The form of this function requires to be experimentally determined and plotted as a curve for certain simple geometrical plan-forms such as the ellipse and the rectangle, the co-ordinates to represent respectively the *aspect ratio* and the corresponding values of κ.

The quantity ϵ is also some function of the aspect ratio, and again we are lacking in experimental information. The values subsequently employed for κ and ϵ are those which the author is in the habit of using, and which are found to give results reasonably near the truth; they have not, however, been determined or verified by any scientific method, and must at present be regarded as open to suspicion.

It has been assumed in the present section that there is no loss of energy incidental to "handling" the fluid other than that due to skin friction. It is in practice possible that there is some unavoidable loss in eddy making by the aerofoil itself; especially is this probable if the angle β is considerable. Apart from the practical considerations introduced by the necessity for *thickness* in the aerofoil, which probably imposes a minimum limit on its fore and aft dimension, there are reasons (which will be discussed hereafter) for supposing that *perfect continuity of motion* is not possible under the conditions of *finite lateral extent*. Whatever defect in the theory may be introduced by considerations of this nature may be legitimately ignored at the present stage.[1]

§ 174. **Gliding Angle.**—Let γ represent, in circular measure, the *gliding angle*—that is, the angle of flight path at which the force to overcome the resistance is exactly provided for by the component of gravity in the path of flight. It will be assumed that γ comes within the definition of a *small angle, i.e.*, $\gamma = \sin\ \gamma = \tan\ \gamma$ with sufficient approximation. Then—

$$\gamma = \frac{\text{Resistance}}{\text{Weight}}.$$

Now weight $(W) = \rho\,\kappa\,A\,V^2\,(a + \beta)$ and resistance comprises—

(1) Aerodynamic resistance $= \dfrac{\text{Energy per sec.}}{\text{Velocity}}$

$$= \frac{\rho\,\kappa\,A\,V^2\,(\beta^2 - a^2)}{2}$$

and

(2) Skin frictional $= \xi\,C\,\rho\,A\,V^2$, and

(3) Body resistance $C\,\rho\,a\,V^2$ where a is a normal plane area to which the body resistance is equivalent.

Now (3) is a superadded resistance with which for the moment we will not concern ourselves, so that we have—

$$\gamma = \frac{\beta^2 - a^2}{2\,(a + \beta)} + \frac{\xi\,C}{\kappa\,(a + \beta)} = \frac{(1 - \epsilon)}{2}\,\beta + \frac{\xi\,C}{(\epsilon + 1)}.$$

But we know that for *Least Resistance* these terms are equal, consequently under the conditions of *Least Gliding Angle* $(= \gamma_1)$ we have $\gamma_1 = (1 - \epsilon)\,\beta_1$, that is to say, the *least gliding angle* will be

$$\gamma_1 = (1 - \epsilon)\,\sqrt{\frac{2\,\xi\,C}{\kappa\,(1 - \epsilon^2)}}.$$

[1] Compare § 189.

If γ_1 be expressed in degrees this will require to be multiplied by $\frac{180}{\pi}$.

§ 175. Taking Account of Body Resistance.—The foregoing investigation has included the temporary assumption that the whole direct resistance is constituted by the skin friction of the aerofoil, as in prop. i. of the preceding chapter. We will now take into account the influence of a resistance independent of the surface of the aerofoil, the body resistance or x_1 of § 165.

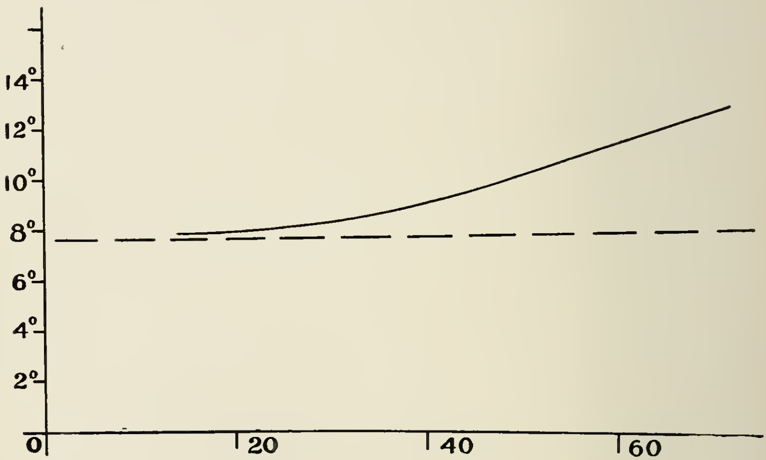


Fig. 111.

We know that such a resistance, which may be represented by an equivalent normal plane, inevitably results in an increase of the gliding angle; also that this increase will be less the lower the velocity, for, according to the equation of the foregoing section (and Prop. V. of § 166), the gliding angle is constant, neglecting body resistance, so long as the aerofoil is properly designed, and does not depend upon the velocity; we may therefore regard the gliding angle as made up of two parts: the part which is constant in respect of velocity, and the part due to body resistance, which varies as the velocity squared.

The resistances, and therefore the gliding angles, may be presented in the form of a diagram (Fig. 111), in which abscissae represent velocity and ordinates the gliding angle; the dotted line represents the constant resistance, and the curve (struck from the dotted line as datum to the equation $\gamma \propto V^2$) shows the manner in which the resistance increases with the velocity. Values of V and γ have been assigned for a supposititious case.

§ 176. Value of β and γ for Least Horse-power.—By prop. ii., § 164, we know that the condition for least horse-power is—
$y = 3x$, when $y = 3x$ let $\beta = \beta_2$.

Then, following § 173—

$$\tfrac{1}{2}\,\kappa\,\beta_2^{\,2}\,(1 - \epsilon^2) = 3\,\xi\,C$$

or

$$\beta_2^{\,2} = \frac{6\,\xi\,C}{\kappa\,(1 - \epsilon^2)}$$

$$\beta_2 = \sqrt{\frac{6\,\xi\,C}{\kappa\,(1 - \epsilon^2)}}.$$

A result that otherwise follows from corollary to prop. iii.—

$$\beta_2 = \sqrt{3}\,\beta_1.$$

Let γ_2 = gliding angle for least horse-power. Following § 174 we have—

$$\gamma_2 = \frac{y + x}{W} \quad \text{where } y = 3\,x$$

or,

$$\gamma_2 = 1\tfrac{1}{3}\,\frac{W}{y} = 1\tfrac{1}{3}\,\frac{\beta_2^{\,2} - a^2}{2\,(\beta_2 + a)}$$

$$\therefore \quad \gamma_2 = \tfrac{2}{3}\,\beta_2\,(1 - \epsilon)$$

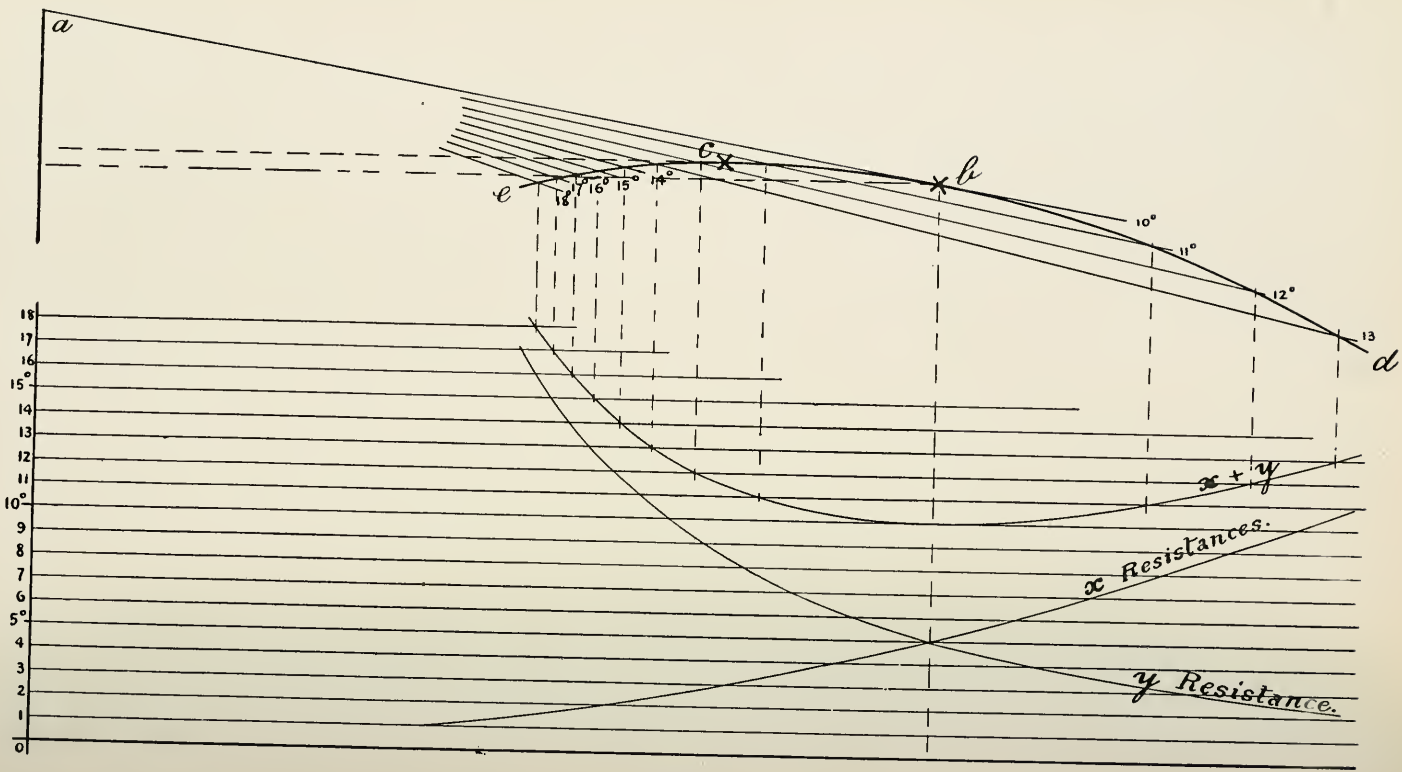
or in terms of β_1—

$$\gamma_2 = \frac{2}{\sqrt{3}}\,\beta_1\,(1 - \epsilon)$$

or,

$$\gamma_2 = \frac{2}{\sqrt{3}}\,\gamma_1 = 1{\cdot}155\,\gamma_1 \text{ (approx.)}.$$

In Fig. 112 the x and y resistances are shown as curves separately and superposed. In the lower portion of the figure



a
b
c
d
e
10°
11°
12°
13
14°
15°
16°
17°
18°
x + y
x Resistances.
y Resistance.
18
17
16
15°
14
13
12
11
10°
9
8
7
6
5°
4
3
2
1
0

FIG. 112.

abscissae represent velocity and ordinates γ values. It is evident that so long as we are confined to the small angle hypothesis the resistances may be thus represented and the sum of the separate γ angles will give the resultant angle.

In the upper part of Fig. 112 the γ angle is represented graphically, it being supposed that the aerodrome is launched from the point *a*.

Thus, if *a b* represent the gliding angle of least resistance (shown for example as 10°) then *a c* will represent the gliding angle for least power, the angle being 11°·55. If we suppose that two aerodromes are launched simultaneously from the point *a* (of equal weight and "sail" area and plan-form), the one being designed for least resistance and the other for least power, their respective trajectories will be the two straight lines *a b* and *a c*, and their positions after the lapse of a certain definite time will be given by the points *b* and *c* where *a b* is to *a c* in the relation $\sqrt[4]{3} : 1$ (prop. iii., § 164). We may draw a curve *e c b d* through the points *c* and *b*, which will represent the position occupied by aerodromes simultaneously launched from the point *a*, for other values of β.

Now since the angle of least resistance is the minimum gliding angle, the line *a b* will be a tangent to the curve *e b d* at the point *b*, and since the least power expenditure corresponds to the slowest rate of fall, the tangent to the curve at the point *c* will be horizontal; we have thus defined the character of the curve in question, which represents the simultaneous *loci* for similar aerodromes of different β values.

The existence or otherwise of *body resistance* does not affect the problem as here presented; it is included in the plotting as one of the resistances that vary as V^2.

§ 177. The Values of the Constants.—The paucity of reliable data has already been made the subject of comment, and the values of many of the constants here given can only be regarded as rough approximations. To prevent misapprehension on this point, the tabulated figures, where not considered reliable, have been entitled "plausible values," and accompanied by a sign of interrogation (?).

It has been demonstrated in § 161 that, on the hypothesis of constant sweep, the constants C c and κ are related to one another according to the equation—

$$\frac{a}{\beta} = \frac{c\,C}{\kappa} - 1,$$

or, employing as before, the symbol ϵ to denote a/β, we have—

$$\epsilon = \frac{c\,C}{\kappa} - 1,$$

that is to say, theory supplies us with a link connecting the whole of the constants involved in the equations of best value of β and least value of γ.

Of the above constants the value of C is known for planes of different aspect ratio from the experiments of Dines, the results being given in the form of a curve in Fig. 89 (Chap. V.). These values tabulated are as follows :—

TABLE I.

Aspect Ratio "n."	Constant "C."
3	·685
4	·70
5	·71
6	·72
7	·725
8	·73
—	—
10	·74
—	—
12	·75

("C" is defined by equation— $P_{90} = C\rho V^2$.)

The values of c are less authentically known; they must be regarded at present merely as *plausible values*[1]; they have been

[1] See footnote on § 172 as to c values.

given plotted for different values of aspect ratio in Figs. 105 and 106, and, tabulated, are as follows:—

TABLE II.

Aspect Ratio "n."	Constant "c."
3	2·16 (?)
4	2·27 (?)
5	2·38 (?)
6	2·48 (?)
7	2·55 (?)
8	2·62 (?)
—	—
10	2·73 (?)
—	—
12	2·80 (?)

Compare Göttingen figures $c = \frac{3(n-1)+1}{n}$ or $c = \frac{3n-1.5}{n}$

§ 178. **On the Constants** κ **and** ϵ.—Of the constant κ we know but little with certainty. Langley's experiments with two pairs of planes four inches by fifteen inches superposed (Fig. 102) suggest that for planes whose aspect ratio is about 4 or 5 κ has a value of somewhat less than 1. For reasons given in § 161 the actual value is probably somewhat greater than that ascertained experimentally for *pairs* of superposed planes.

On the value of ϵ we are entirely without information so far as direct experiment is concerned. If the value of κ were known for an aerofoil of given aspect ratio, the value of ϵ can be obtained from the equation given in the preceding article, *i.e.*, $\epsilon = \frac{c\,C}{\kappa} - 1$.

We may provisionally assume that κ is a function of n and constant in respect of other variables. It is true that we have taken no account of the influence of *plan-form*, but we may legalise our position in this respect by specifying some standard form such as a rectangle, and leave the onus of drawing up tables of equivalent proportions in any other form to future experimenters.

At present the quantitative data are in so unsatisfactory a state that it is almost unnecessary to specify the precise form to which n values are supposed to relate; we may take it that we are dealing with a rectangular plan-form, and that n denotes the lateral breadth in terms of the fore and aft dimension; thus, for planes in pterygoid aspect n has a value greater than unity, and for planes in apteroid aspect, less.

Now if κ is a function of n alone, ϵ is also a function of n

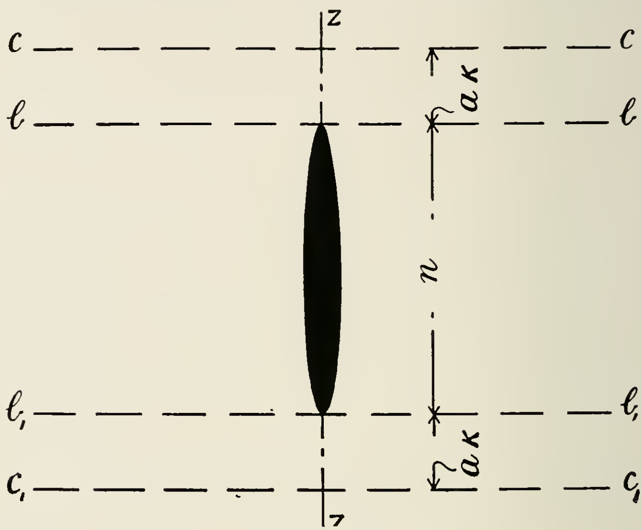


FIG. 113.

alone, and if we can by experiment or theory establish a form of expression in the one case, the other follows from the equation.

It is evident that the circumstances determining ϵ are foreign to our present hypothesis, and we shall require to temporarily take our stand outside this hypothesis in order to investigate the question.

§ 179. An Auxiliary Hypothesis.—Let us suppose an aerofoil represented in plan in Fig. 113 supported in a continuous medium; then if M_a be the upward momentum communicated to the air passing between planes represented by the lines b b and b_1 b_1 at the time when its upward momentum is a maximum—that is, when it comes within the direct influence of the aerofoil (the descending field of Chap. IV.); then, assuming a field of force symmetrical about the plane $z\ z$, M_α will also represent the upward momentum communicated to the air in partially arresting its downward motion during *recess*.

And the air outside the surfaces $b\ b$ and $b_1\ b_1$ will be receiving upward momentum the whole time it is in the field. Let the sum of this momentum be denoted by M_1.

Now, since the total residuary momentum must be zero (§§ 5 and 117) the downward momentum remaining in the air between the surfaces $b\ b$ and $b\ b$ is also equal to M_1, and if M_β be the downward momentum in the air when it quits the descending field we shall have: $M_\beta = M_\alpha + M_1$.

But according to the main hypothesis we may represent $\frac{M_\alpha}{M_\beta}$ by $\frac{\alpha}{\beta}$, that is

$$\epsilon = \frac{M_\alpha}{M_\beta} = \frac{M_\alpha}{M_\alpha + M_1}. \qquad (1)$$

It remains for us to assess the value of M_1 in terms of M^β.

Let us suppose that, in a manner analogous to the limitation of the sweep, the air external to the surfaces $b\ b\ b_1\ b_1$ be represented by the limited region cut off by two further surfaces $c\ c$ and $c_1\ c_1$; then it is evident that the distance separating these surfaces will be greater the greater the lateral extension of the aerofoil.

Calling the fore and aft dimension of the aerofoil unity, so that its lateral dimension will $=$ n, *let us assume* that the distance between $b\ b$ and $c\ c$ is proportional to κ, and let it be denoted by $a\ \kappa$. We have no direct means of testing the accuracy of this assumption; we can only say that it is a reasonable assumption, since the conditions that influence the depth of the layer of air acted upon obviously affect the extent of the disturbance of the fluid in other directions.

Then the upward momentum received by the air at the time of its crossing the plane $z\ z$ is, from considerations of field symmetry, just half the total eventually imparted, that is $= \frac{M_1}{2}$, but we are supposing that the air that reaches the plane $z\ z$ between $b\ b$ and $c\ c$ has received momentum *pro rata* with that within the region bounded by $b\ b\ b_1\ b_1$, that is to say :—

$$M_a \Big/ \frac{M_1}{2} = \mathrm{n} \Big/ 2\,a\,\kappa,$$

or
$$M_1 = M^a \frac{4\,a\,\kappa}{\mathrm{n}}$$

that is, by (1)
$$\epsilon = \frac{M_a}{M_a\left(1 + \frac{4\,a\,\kappa}{\mathrm{n}}\right)}$$

or
$$\epsilon = \frac{\mathrm{n}}{\mathrm{n} + 4\,a\,\kappa}.$$

We may take a constant e to represent $4a$, and our expression is—

$$\epsilon = \frac{\mathrm{n}}{\mathrm{n} + e\,\kappa}.$$

§ 180. κ and ϵ, Plausible Values.—We are now able to find an expression for κ in terms of c, C, and n, for we already have the equation—

$$\epsilon = \frac{c\,C}{\kappa} - 1, \quad \therefore \quad \frac{\mathrm{n}}{\mathrm{n} + e\,\kappa} = \frac{c\,C}{\kappa} - 1.$$

or
$$c\,C = \frac{2\,\kappa\,\mathrm{n} + e\,\kappa^2}{\mathrm{n} + e\,\kappa}$$

$$\therefore \quad c\,C\,\mathrm{n} + c\,C\,e\,\kappa = 2\,\kappa\,\mathrm{n} + e\,\kappa^2$$

$$\therefore \quad e\,\kappa^2 + 2\,\mathrm{n}\,\kappa - c\,C\,e\,\kappa = c\,C\,\mathrm{n}$$

or,
$$\kappa^2 + \left(\frac{2\,\mathrm{n}}{e} - c\,C\right)\kappa = \frac{c\,C\,\mathrm{n}}{e},$$

whence—
$$\kappa = \pm\sqrt{\left\{\frac{c\,C\,\mathrm{n}}{e} + \left(\frac{\mathrm{n}}{e} - \frac{c\,C}{2}\right)^2\right\}} - \frac{\mathrm{n}}{e} + \frac{c\,C}{2},$$

the rest is a matter of choosing such a value of e as fits in best with experience. The author has taken $e = 3{\cdot}3$, and this is the basis on which the following Table of plausible values of κ and ϵ is founded.

§ 181. Best Value of β. Least Value of γ.—Assuming the values given in the Tables (I., II. and III.), we are now in a position to obtain numerical values for the best values of the angle β for aerofoils of different aspect value. Table IV. illustrates the process of calculation in the case of the pterygoid

TABLE III.

Plausible Values, κ and ϵ.

n.	κ.	ϵ.
3	1·00 (?)	·48 (?)
4	1·03 (?)	·54 (?)
5	1·064 (?)	·59 (?)
6	1·10 (?)	·62 (?)
7	1·12 (?)	·65 (?)
8	1·14 (?)	·68 (?)
—	—	—
10	1·175 (?)	·72 (?)
—	—	—
12	1·195 (?)	·75 (?)

TABLE IV.

β values for γ = minimum.

Calculated from Equation, $\beta_1 = \sqrt{\dfrac{2\,\xi\,C}{\kappa\,(1 - \epsilon^2)}}$ for ξ = ·03.

n.	κ (?)	ϵ(?).	ϵ^2.	$1-\epsilon^2$.	$\kappa(1-\epsilon^2)$.	C.	ξ.	$2\,\xi\,C$.	β_1^2.	β_1.	$\beta_1°$.
3	1·00	·48	·23	·770	·770	·685	·03	·0411	·0534	·231	13·2°
4	1·03	·54	·291	·709	·730	·700	·03	·0420	·0575	·240	13·75°
5	1·064	·59	·348	·652	·695	·71	·03	·0426	·0612	·247	14·14°
6	1·10	·62	·384	·616	·678	·72	·03	·0432	·0638	·252	14·4°
7	1·12	·65	·422	·578	·648	·725	·03	·0435	·0671	·259	14·8°
8	1·14	.68	·462	·538	·614	·73	·03	·0438	·0715	·268	15·0°
—	—	—	—	—	—	—	—	—	—	—	—
10	1·175	.72	·518	·482	·567	·74	·03	·0444	·0783	·280	16·0°
—	—	—	—	—	—	—	—	—	—	—	—
12	1·195	·75	·562	·438	·523	·75	·03	·0450	·0861	·293	16·8°

aerofoil ξ being taken as = ·03. Table V. gives the results for values of ξ equal ·025, ·020, ·015 and ·010 in the respective columns; both Tables IV. and V. being values of β appropriate to *least gliding angle*, that is, for γ = minimum.

Table VI. gives the theoretical values of γ corresponding to

TABLE V.

Values of β_1 *(continued).*

n.	$\xi = \cdot 025$	$\xi = \cdot 020$	$\xi = \cdot 015$	$\xi = \cdot 010$
3	·210 = 12·0°	·189 = 10·8°	·163 = 9·3°	·133 = 7·6°
4	·219 = 12·5°	·196 = 11·2°	·169 = 9·7°	·138 = 7·9°
5	·226 = 12·9°	·202 = 11·6°	·174 = 10·0°	·142 = 8·1°
6	·230 = 13·2°	·206 = 11·8°	·178 = 10·2°	·145 = 8·3°
7	·236 = 13·5°	·212 = 12·15°	·183 = 10·5°	·149 = 8·5°
8	·244 = 14·0°	·218 = 12·5°	·189 = 10·8°	·154 = 8·8°
—	—	—	—	—
10	·256 = 14·7°	·228 = 13·0°	·198 = 11·3°	·161 = 9·2°
—	—	—	—	—
12	·268 = 15·3°	·239 = 13·7°	·207 = 11·8°	·169 = 9·7°

TABLE VI.

Least Gliding Angle $(= \gamma_1)$ *(Theoretical).*

Minimum value of γ or $\gamma_1 = (1 - \epsilon) \sqrt{\dfrac{2 \xi C}{\kappa (1 - \epsilon^2)}}$

n.	$\xi = \cdot 030$		$\xi = \cdot 025$		$\xi = \cdot 02$		$\xi = \cdot 015$		$\xi = \cdot 010$	
3	6·85°	1 : 8·3	6·25°	1 : 9·2	5·6°	1 : 10·2	4·8°	1 : 12	3·95°	1 : 14·5
4	6·32°	1 : 9	5·75°	1 : 10	5·15°	1 : 11·1	4·4°	1 : 13	3·65°	1 : 15·7
5	5·8°	1 : 10	5·3°	1 : 10·8	4·75°	1 : 12	4·1°	1 : 14	3·40°	1 : 16·8
6	5·5°	1 : 10·4	5·0°	1 : 11·5	4·5°	1 : 12·8	3·9°	1 : 14·7	3·20°	1 : 17·9
7	5·2°	1 : 11	4·7°	1 : 12·2	4·25°	1 : 13·5	3·6°	1 : 15·9	3·00°	1 : 19·1
8	4·8°	1 : 12	4·5°	1 : 12·8	4·0°	1 : 14·4	3·4°	1 : 16·8	2·80°	1 : 20·5
—	—	—	—	—	—	—	—	—	—	—
10	4·5°	1 : 12·8	4·1°	1 : 14	3·65°	1 : 15·8	3·2°	1 : 17·9	2·60°	1 : 22
—	—	—	—	—	—	—	—	—	—	—
12	4·2°	1 : 13·7	3·8°	1 : 15	3·42°	1 : 16·8	3·0°	1 : 19	2·40°	1 : 23·9

the ascertained values of β, that is to say, this Table represents the theoretical minimum gliding angle for the values of ξ employed.

In Tables IV. and V. the angle is given both in circular measure and in degrees; in the case of the gliding angle Table VI. the equivalent is given in the inverse form, *i.e.*, as a *gradient*. In all cases the assumption is that of the *small angle* as already stated.

In actual aerodrome models, owing to the necessity for organs of equilibrium the resistance is greater than that due to the considerations taken into account in the foregoing Table; there is

TABLE VII. (AEROPLANE).

Least Resistance. Values of $\beta_1°$.

From Equation— $\beta_1° = \frac{180}{\pi}\sqrt{\frac{\xi}{c}}$.

Values of c assumed from Table II. (*Plausible Values*).

n.	β for values of ξ as follows:— ·030	·025	·020	·015	·0125	·010
3	6·76°	6·16°	5·51°	4·77°	4·35°	3·90°
4	6·60°	6·02°	5·38°	4·66°	4·25°	3·80°
5	6·43°	5·87°	5·25°	4·55°	4·15°	3·71°
6	6·30°	5·76°	5·15°	4·45°	4·07°	3·64°
7	6·22°	5·67°	5·07°	4·39°	4·00°	3·58°
8	6·13°	5·59°	5·00°	4·33°	3·95°	3·54°
—	—	—	—	—	—	—
10	6·00°	5·48°	4·90°	4·24°	3·87°	3·46°
—	—	—	—	—	—	—
12	5·93°	5·41°	4·84°	4·19°	3·83°	3·42°

additional resistance due to the added surface, or *body resistance*. Owing to this and other causes which will be explained later, the gliding angle is never found to be as low as the theoretical value, and in the most carefully made model is usually *at least* 50 % greater than theory would indicate.

Reverting to the simple case of the plane aerofoil, or aeroplane (§§ 162 and 172), we have seen that the value of β for least resistance, that is to say, least gliding angle, is given by the expression $\beta_1 = \sqrt{\frac{\xi}{c}}$, and since the aerodynamic and direct resistances are equal we have least value of $\gamma = 2\sqrt{\frac{\xi}{c}}$. In Tables VII. and VIII. the calculated angles are given for values of ξ ranging from ·01 to ·03.

§ 182. The Aeroplane. Anomalous Value of ξ.—The actual behaviour of an aeroplane presents an anomaly with regard to

TABLE VIII. (AEROPLANE).

Theoretical Least Gliding Angles. ($\gamma = min.$).

n.	ξ = ·030	·025	·020	·015	·0125	·010
3	13·52°	12·32°	11·02°	9·54°	8·70°	7·80°
4	13·20°	12·04°	10·76°	9·32°	8·50°	7·60°
5	12·86°	11·74°	10·50°	9·10°	8·30°	7·42°
6	12·60°	11·52°	10·30°	8·90°	8·14°	7·28°
7	12·44°	11·34°	10·14°	8·78°	8·00°	7·16°
8	12·26°	11·18°	10·00°	8·66°	7·90°	7·08°
—	—	—	—	—	—	—
10	12·00°	10·96°	9·8°	8·48°	7·74°	6·92°
—	—	—	—	—	—	—
12	11·86°	10·82°	9·68°	8·38°	7·66°	6·84°

the value of ξ. It has been remarked in the previous section that in the case of the pterygoid aerofoil the theoretical results can never in practice be fully realised, owing partly to the necessity of added surface and partly to other causes. In the case of the aeroplane, in spite of the fact that the same values of ξ are employed in the calculation the reverse is the case, and investigation shows that *in effect* the value of ξ is considerably less in the case of an aeroplane than its ordinary value, and may amount to no more than half the coefficient as ordinarily found applicable.[1] The result is that the aeroplane shows results far better than theory would indicate unless a diminished value of ξ be employed in the equation.

The probable explanation of this anomaly is to be found in the supposition that for the angles investigated the flow is of the Rayleigh-Kirchhoff type, as illustrated diagrammatically in Fig. 98 (a), the result being that the effect of skin friction is only felt on the one face of the plane instead of on both faces, as would be the case if the flow were conformable, and consequently the apparent value of ξ is only about half its real value.

There is a serious but not insuperable difficulty attached to the foregoing explanation. It would appear that since the "dead water" is itself subject to a tangential drag at its free surfaces, and since, as a whole, it has no influence to keep it in position other than the reaction of the aeroplane itself, this frictional drag must be transmitted to the aeroplane, and so in some way take the place of the missing skin friction.

On examining the matter in greater detail, it is evident that the form of the dead water region is determined primarily by the dynamics of the live stream, and if the fluid be supposed frictionless the dead water will extend indefinitely rearward, and its pressure will throughout be uniform. If now we take into account the effects of viscosity there will be a frictional or viscous drag acting tangentially at the surface of discontinuity between the dead water and the live stream, and referring to Fig. 114, it is evident that the cumulative effect of this drag will be to create a *pressure gradient*, the pressure at A being less than that at B, and that at B less than that at C, and so on. In consequence of this pressure difference the dead water will become

[1] The results so far obtained by the author on the value of ξ by different methods are not altogether in harmony. Since writing this and the following sections (§§ 182, 183, and 184), certain experiments made with a new instrument, the *aerodynamic balance* (Ch. X.), seem to indicate that the conclusions here formulated may require qualifying; the results at present available, however, are not conclusive, and it has been thought best to present the argument in its original form.

the seat of a lively circulation as indicated by the arrows, the motion of the fluid in the vicinity of the plane being in the direction of flight, and that in the vicinity of the free surface being in the opposite direction. Now the result of this will be to produce a tangential drag in a *forward direction ;* in fact, any skin friction experienced on the upper face or "back" of the plane will be of *negative sign ;* we are thus unable to attribute the "retention" of the dead water to the direct influence of the plane.

On following the matter further it is evident that it is the partial vacuum in and about the region *a a a* that supplies the necessary

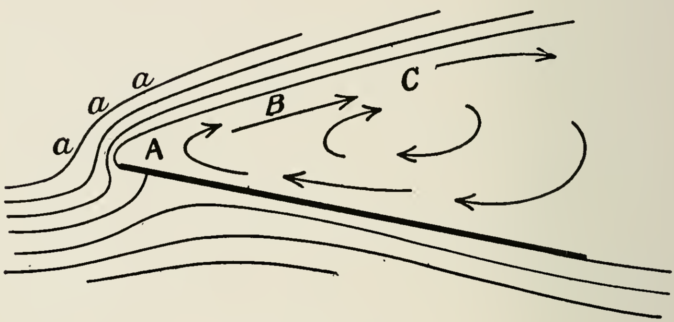


Fig. 114.

reaction to prevent the dead water from being washed away, the lines of flow being at this point in close proximity to one another, as indicated in the figure. We thus find that the back of the plane is not only apparently, but is *really*, relieved of the frictional drag, which is actually borne in some way dynamically by the fluid itself.

§ 183. Aeroplane Skin Friction. Further Investigation.—The present stage of our explanation cannot be regarded as entirely satisfactory. It would appear to be essential, if we suppose the aeroplane to be maintained in steady motion by an applied force, that all reactions experienced by the fluid must be eventually traceable to the applied force. In the case under consideration we have traced the action and reaction merely from one part of the live stream to another part. It remains to be shown in what manner the motion of the fluid in the region *a a a* (Fig. 114) is *counterpart to* some component resistance in the line of flight not otherwise essential.

Let us suppose a limited stratum of fluid to be dealt with, firstly, by a system of superposed aeroplanes; and secondly, by a kind of honeycomb of curved tubes whose leading orifices point in the direction of flight, and whose trailing or *discharge* orifices make the same angle with the line of flight as the angle β of the aeroplane system. Then, if W be the total weight supported in either case, the resistance in the case of the aeroplanes will be (for small values of β) $= W\beta$, but in the case of the curved tubes it will only amount to half this quantity, or $= W\frac{\beta}{2}$, the operating surfaces in either case being supposed frictionless.

It is therefore evident that the aeroplane involves twice as great a resistance to traction as that aerodynamically necessary,[1] and from what we know of the Kirchhoff form of flow we can see that this added traction is employed in generating and maintaining the *spurting forward* of the fluid round the leading edge, indicated by the lines *a a a a* in the figure. When the work expended in traction is entirely devoted to diverting the stream, as in the theoretical case of the curved pipe system, then there is no spurting forward of the fluid, and no discontinuity in the system of flow; and on the other hand, the operating surfaces are fully exposed to frictional resistance. When the stream is brusquely diverted by an aeroplane there is an aerodynamic resistance involved in excess of that necessary to divert the flow, and this, by giving rise to a form of flow of the discontinuous type, diminishes the frictional resistance.

Owing in part to the return current in the dead water region, and in part to the forward motion of the fluid on the front face

[1] This is on the basis of ignoring the cyclic reaction; if this be taken into account the aeroplane is at a still greater disadvantage.

of the plane in the vicinity of its leading edge, and again in part to the slowing of the flow over the remainder of the face (owing to its being a *pressure region*), it would appear that the net skin friction might even be less than that computed on the basis of a single surface.

§ 184. Some Consequences of the foregoing Aeroplane Theory.— The consequences of the peculiar behaviour of the aeroplane in respect of skin friction are of considerable moment.

The aeroplane, thanks to its power of evading a considerable portion of the resistance due to skin friction, is capable of being utilised for the support of the load without any very great loss of efficiency. Considered thus, and compared to an aerofoil of pterygoid form, it is found to give results that are really remarkable. Experimenting on a small scale, it is difficult to construct a model with a pterygoid aerofoil that, so far as gliding angle is concerned, will perform better than a ballasted aeroplane of the most crude description. An analogous example is found in the case of the screw propeller. Most of the theory relating to the aeroplane, wing form, and peripteral motion, finds its analogue in the theory of the screw propeller (Chap. IX.), and it is well known to designers of the latter that, so long as the pitch is rightly chosen in view of the torque and thrust, and provided that the angle, area, and proportions, of the blades are suitable, there is but a moderate gain in efficiency to be obtained by departure from the simple helical form of blade.

It is probable that the relative advantage of the pterygoid form becomes greater when the size of the aerodrome is increased, owing to the relations of weight and area discussed in § 196, and the relatively less importance of skin friction. If this should prove to be the case the present theory would account for the remarkable difference between the flight and wing form of birds and insects, showing in detail that which was anticipated in Chap. II. (Compare § 196.) In general the wings of flies, dragon flies, moths, etc., are approximately flat—they are in fact aeroplanes;

and further, when flexed by the pressure to which they are subjected in flight it is probable that they actually present a convex surface to the "wind." On the contrary, the wings of birds are always concave on the under side and convex above; they are in fact true pterygoid forms. This is not only the case when the wing is quiescent but is visibly the case when the bird is in flight. It is of particular interest that some of the larger butterflies and moths—for example, many of the *ornithoptera*—show clearly a rudimentary development of the dipping front edge, proving that this feature is not merely an incident of a different method of construction.

§ 185. The Weight per Unit Area as related to the Best Value of β.—We may now resume the main subject from the point to which it was carried in § 181, and we can show that the value of β corresponding to a minimum gliding angle denotes a definite relationship between the area A, the velocity V, and the load carried W.

According to the hypothesis of constant sweep, we know that the mass dealt with per second is given by the expression $\rho\, k\, A\, V$, and the velocity of the up-current is $\alpha\, V$, and that of downward discharge $= \beta\, V$, on the assumption that we are dealing with small angles.

Consequently the weight supported (W) which is equal to the momentum communicated per second, will be $\rho\, \kappa\, A\, V^2\, (\alpha + \beta)$; but we have $\alpha = \epsilon\, \beta$, so that our expression becomes—

$$W = \rho\, \kappa\, A\, V^2\, \beta\, (\epsilon + 1),$$

or—

$$\frac{W}{A\, V^2} = \rho\, \kappa\, \beta\, (\epsilon + 1), \text{ which is constant.}$$

Now $\frac{W}{A\, V^2}$ may be written $\frac{P_2}{V^2}$ where P_2 denotes pressure, *i.e.*, weight per unit area sustained by the aerofoil. In Table IX. are given values of $\frac{P_2}{V^2}$ for aerofoil of pterygoid form and of different aspect ratio, calculated from values of β given in

Tables IV. and V., for ξ taken as ·03, ·025, ·02, ·015, and ·010 in the respective columns.

TABLE IX.

PTERYGOID AEROFOIL.

Values of P_2/V^2 for Least Resistance.

n.	$\kappa\rho(\epsilon+1)$.	P_2/V^2 for values of ξ as follows:—				
		ξ = ·03	ξ = ·025	ξ = ·02	ξ = ·015	ξ = ·010
3	·1154	·0266	·0242	·0218	·0188	·0153
4	·1236	·0296	·0270	·0242	·0209	·0170
5	·1319	·0326	·0298	·0266	·0230	·0188
6	·1390	·0350	·0320	·0286	·0247	·0204
7	·1440	·0372	·0340	·0305	·0263	·0215
8	·1493	·0400	·0364	·0326	·0283	·0231
—	—	—	—	—	—	—
10	·1574	·0440	·0403	·0359	·0311	·0254
—	—	—	—	—	—	—
12	·1630	·0477	·0437	·0390	·0337	·0275

In the employment of this Table it must be remembered that we have to deal with *British absolute units*, so that P_2 will be *poundals* per square foot; thus, supposing it were desired to design an aerofoil of aspect ratio n = 10 to travel at a velocity of 40 feet per second, then the value of P_2 for least resistance will be = 1,600 × ·0440 (taking ξ = ·03), or 70·4 *poundals* or 2·2 lbs. per square foot (approximately).

In Table X. the appropriate load per square foot is given for velocities from 10 to 80 feet per second for various values of "n," the value of ξ has been taken as ·03, ·02, and ·01.

§ 186. Aeroplane Loads for Least Resistance.—The pressure, or load, per unit area that an *aeroplane* will economically sustain is considerably less than that tabulated in the preceding section for aerofoils of pterygoid form.

Table X.

Pterygoid Aerofoil.

Load (pounds) per Square Foot for Least Resistance.

	Ft. per Sec.	Values of "n" :— 3.	4.	5.	6.	7.	8.	10.	12.
ξ = ·030.	5	·020	·023	·025	·027	·029	·031	·034	·037
	10	·082	·092	·101	·108	·115	·124	·136	·148
	15	·186	·207	·228	·245	·260	·280	·307	·333
	20	·331	·367	·405	·435	·462	·497	·546	·593
	25	·516	·574	·633	·680	·721	·777	·852	·926
	30	·743	827	·911	·978	1·04	1·12	1·23	1·33
	35	1·01	1·12	1·24	1·33	1·41	1·52	1·67	1·82
	40	1·32	1·47	1·62	1·74	1·84	1·99	2·18	2·37
	50	2·06	2·30	2·53	2·71	2·88	3·11	3·41	3·70
	60	2·97	3·31	3·65	3·91	4·15	4·47	4·91	5·33
	70	4·05	4·50	4·96	5·32	5·64	6·09	6·70	7·25
	80	5·29	5·89	6·48	6·95	7·39	7·95	8·73	9·48
ξ = ·020.	5	·017	·018	·020	·022	·023	·025	·028	·030
	10	·068	·075	·082	·089	·094	·101	·111	·121
	15	·152	·169	·186	·200	·213	·228	·250	·272
	20	·270	·300	·330	·355	·379	·405	·445	·484
	25	·390	·433	·475	·511	·545	·582	·641	·697
	30	·610	·676	·743	·800	·852	·911	1·00	1·09
	35	·830	·920	1·01	1·08	1·16	1·24	1·36	1·48
	40	1·08	1·20	1·32	1·42	1·51	1·62	1·78	1·94
	50	1·69	1·88	2·06	2·22	2·37	2·53	2·78	3·03
	60	2·44	2·70	2·97	3·20	3·40	3·64	4·01	4·36
	70	3·32	3·68	4·05	4·35	4·64	4·96	5·46	5·93
	80	4·33	4·81	5·30	5·70	6·07	6·47	7·13	7·75
ξ = ·010.	5	·012	·013	·014	·015	·016	·018	·019	·021
	10	·047	·053	·058	·063	·066	·071	·079	·085
	15	·107	·119	·131	·142	·150	·161	·177	·192
	20	·190	·211	·234	·253	·267	·287	·315	·341
	25	·298	·331	·366	·396	·418	·450	·495	·535
	30	·427	·475	·526	·570	·601	·645	·710	·769
	35	·582	·647	·717	·777	·820	·880	·967	1·04
	40	·760	·845	·935	1·01	1·07	1·15	1·26	1·36
	50	1·18	1·32	1·46	1·58	1·67	1·79	1.97	2·13
	60	1·71	1·90	2·10	2·28	2·40	2·58	2·84	3·07
	70	2·32	2·58	2·86	3·10	3·27	3·51	3·86	4·18
	80	3·04	3·38	3·73	4·05	4·27	4·59	5·05	5·45

For the aeroplane we know that the weight supported per unit area is (for small angles) given by the expression—$c\,\beta \times C\,\rho\,V^2$, which is $= P_3$.

We therefore have $\frac{P_3}{V^2} = \rho\,c\,C\,\beta_1$.

For values of ξ respectively ·02, ·015, ·0125, and ·01, and taking ρ as before = ·078, the values of $\frac{P_3}{V^2}$ for least resistance are given in Table XI.

TABLE XI.

AEROPLANE.

Values of P_3/V^2 for Least Resistance.

n.	ξ = ·020	ξ = ·015	ξ = ·0125	ξ = ·010
3	·0111	·0096	·0087	·0078
4	·0116	·0100	·0091	·0082
5	·0121	·0104	·0095	·0085
6	·0124	·0108	·0098	·0088
7	·0127	·0110	·0100	·0090
8	·0130	·0112	·0102	·0092
—	—	—	—	—
10	·0135	·0116	·0106	·0095
—	—	—	—	—
12	·0138	·0119	·0109	·0098

In Table XII. the foregoing results have been interpreted as pounds per square foot for different values of V ranging from 5 to 80 feet per second, and for values of ξ = ·02, ·015, and ·01.

It is scarcely necessary to remark that the values given in the preceding Tables are not based on sufficiently reliable data to justify their being carried to so many places of decimals; the figures as tabulated have a *probable error* of 10 per cent. or even 20 per cent. one way or the other, and the employment of the third significant figure is only justified as a means of showing the relation of any one value to those adjacent to it in the Table.

It has not been thought necessary to re-tabulate in metric

Table XII.—Aeroplane Data.

Pounds per square foot for different values of V for Least Resistance.

	Feet per Second.	Values of "n" :—							
		3.	4.	5.	6.	7.	8.	10.	12.
ξ = ·020.	5	·0086	·0090	·0094	·0096	·0098	·0101	·0104	·0107
	10	·034	·036	·037	·038	·039	·040	·042	·043
	15	·077	·081	·084	·086	·088	·090	·094	·096
	20	·138	·144	·150	·154	·158	·161	·167	·171
	25	·215	·225	·234	·240	·246	·252	·261	·268
	30	·310	·322	·337	·346	·354	·363	·376	·385
	35	·421	·440	·460	·471	·482	·494	·510	·525
	40	·550	·576	·601	·616	·631	·646	·670	·686
	50	·861	·900	·940	·963	·985	1·01	1·04	1·07
	60	1·24	1·29	1·35	1·38	1·42	1·45	1·50	1·54
	70	1·68	1·76	1·83	1·88	1·93	1·97	2·05	2·10
	80	2·20	2·30	2·40	2·46	2·52	2·59	2·68	2·74
ξ = ·015.	5	·0075	·0078	·0081	·0084	·0086	·0087	·0090	·0093
	10	·0300	·0313	·0325	·0336	·0344	·0350	·0362	·0372
	15	·067	·070	·073	·075	·077	·079	·081	·083
	20	·120	·125	·130	·134	·137	·140	·145	·149
	25	·187	·195	·203	·210	·214	·218	·226	·232
	30	·269	·282	·292	·302	·309	·315	·326	·335
	35	·367	·384	·398	·410	·420	·430	·444	·456
	40	·480	·501	·520	·538	·549	·560	·580	·596
	50	·750	·783	·813	·840	·858	·875	·905	·930
	60	1·08	1·12	1·17	1·21	1·23	1·26	1·30	1·34
	70	1·47	1·53	1·59	1·65	1·68	1·71	1·77	1·82
	80	1·92	2·00	2·08	2·15	2·20	2·24	2·32	2·38
ξ = ·010.	5	·0060	·0063	·0066	·0068	·0070	·0071	·0073	·0076
	10	·024	·025	·026	·027	·028	·028	·029	·030
	15	·054	·057	·059	·061	·063	·064	·066	·068
	20	·097	·102	·105	·109	·112	·114	·118	·122
	25	·151	·159	·165	·171	·175	·179	·185	·191
	30	·242	·254	·264	·273	·279	·286	·295	·304
	35	·297	·312	·323	·335	·342	·350	·361	·373
	40	·387	·407	·422	·437	·447	·457	·472	·487
	50	·606	·637	·660	·683	·699	·714	·738	·761
	60	·871	·916	·950	·984	1·00	1·02	1·06	1·09
	70	1·18	1·24	1·29	1·34	1·37	1·40	1·44	1·49
	80	1·55	1·63	1·69	1·75	1·79	1·83	1·89	1·95

units, for the pressures in kilos per square metre can be obtained with a sufficient degree of approximation by multiplying by five the figures given in Tables X. and XII.

§ 187. Comparison with Actual Measurements.—The portent of the preceding sections may be illustrated by a few examples from Nature.

The herring gull, according to the system of measurement adopted by the author and subsequently explained, carries its load at the rate of about 1·3 to 1·4 pounds per square foot; its n value is 7. Referring to Table X. ($\xi = \cdot 02$) we find this load corresponds to 38 feet per second or about 26 miles per hour, which is probably a fair approximation to its actual speed.

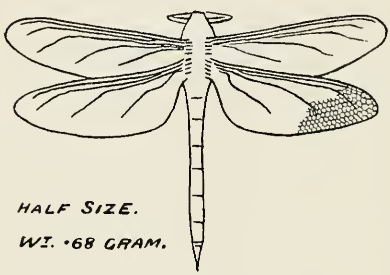


FIG. 115.

The albatros carries about 3 pounds per square foot, and has an **n** value of 12; referring to the Table we find the corresponding velocity to be about 50 feet per second or slightly over 34 miles per hour, which again is probably not far from the truth.

If now we take the case of a dragon-fly: an example weighed and measured by the author (Fig. 115) gave a result, from a planimeter measurement of the whole wing surface, of ·68 grammes on 3·5 square inches, which is ·062 pounds per square foot. The "**n**" value may fairly be taken as about = 4.

Referring to the Tables and taking $\xi = \cdot 02$, we have for *pterygoid form* the corresponding velocity = 9·1 feet per second, or according to Table XII., considering the wings as *planes*, the velocity should be from 13 to 15·6 feet per second, according as ξ is taken as ·02 or ·01.

Unfortunately no scientific measurement of the flight of this insect appears to be available, but its velocity is certainly nearer the latter than the former estimate.

In the case of birds such as those above cited, the soaring mode of flight is so extensively employed that without doubt the process of natural selection, or whatever other method Nature may employ, may be relied upon to have approximated the proportions of least resistance proper to the ordinary velocity of flight of the species. In the case of smaller birds or insects, such as the dragon-fly cited, it is an open question to what extent the problem is modified by the exigencies of active flight, and so the evidence, as confirming or otherwise the present theory, is at the best inconclusive.

§ 188. Considerations Relating to the Form of the Aerofoil.—We have so far specified the form of the aerofoil only so far as the angles α and β are concerned, and have now not only to discuss the other attributes of the fore and aft section, but also the plan form of the aerofoil and its variation of section from point to point, and in addition the shape it presents when viewed along the axis of flight.

Many of the influences at work to affect the form of the aerofoil do not belong to the province of aerodynamics. The question of form, as viewed along the axis of flight, is governed almost entirely by *aerodonetic* considerations, and the discussion of this point will therefore be reserved.

The present subject has already been examined in Chap. IV., § 120; it remains for us now to continue the discussion in the light of the present theory.

If we suppose, provisionally, that the aerofoil section is of the form of the arc of a circle, then such a form would manifestly carry out the requirements of hypothesis with a uniform distribution of pressure on its surface, for we are supposing that the "fluid" consists of a limited layer composed of a number of strata whose individual continuity is preserved, after the manner of Fig. 108. If we suppose such an aerofoil to be gliding in a frictionless fluid, then its trailing and leading edges will be at the same level, for owing to considerations of symmetry it is then that the reaction is vertical. Under these conditions we see (Fig. 116) that the gliding angle will be $= \frac{\beta - a}{2}$.

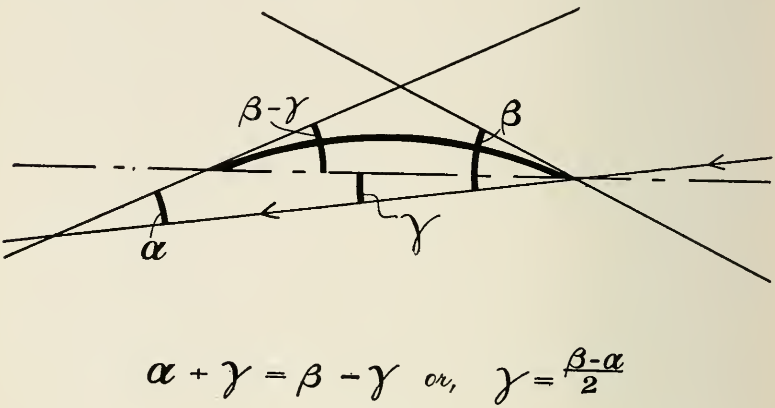


FIG. 116.

This result may be demonstrated more generally for, x resistances being absent,— $\gamma = \frac{y}{W}$

$$= \frac{\beta^2 - a^2}{2(\beta + a)} = \frac{\beta - a}{2}.$$ Compare §§ 174, 176.

Now the ratio of the angles a and β does not depend upon the leading and trailing angles given to the aerofoil, but upon the aspect ratio, so that the design of the aerofoil requires to conform to the ratio so imposed. If we take an aerofoil of arc section there is a particular direction in which it must be propelled in order that it should fulfil the necessary condition, and this direction is in practice determined by a *directive organ* which usually takes the form of a *tail plane*.

Let us examine the effect of an incorrect adjustment of the directive organ; that is to say, we will examine the effect of *incorrectly designing the aerofoil* in respect to the value of ϵ $(= a/\beta)$. Firstly, suppose it be adjusted so that the "dip" of the front edge is insufficient, then the up-current will no longer strike the edge conformably, and, in the case of a real fluid, a discontinuity will result, as illustrated in Fig. 117; such a discontinuity may be a trivial matter involving only a small *pocket* of "dead water" (*a*), or it may be more serious so that the form of flow resembles that generated by an aeroplane (*b*); in either case we

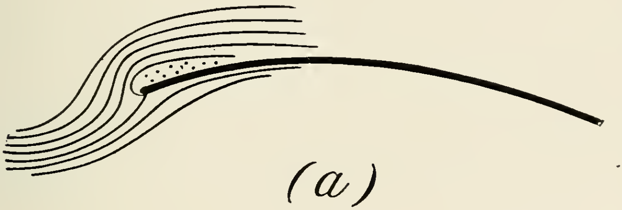


Fig. 117a.

know, from the great efficiency obtainable from the aeroplane, that the effects are not disastrous.

If, secondly, we suppose that the leading edge has too much

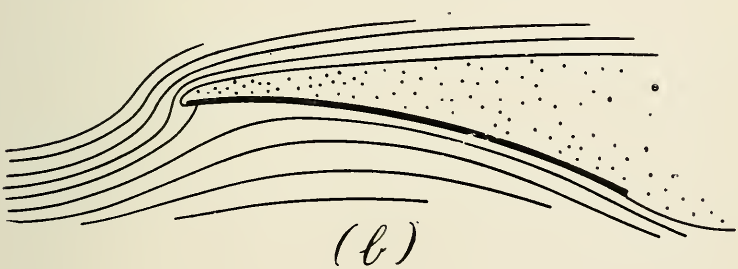


Fig. 117b.

"dip," the want of conformity is in the opposite direction, and the surface of discontinuity springs from beneath the leading edge as depicted in Fig. 118; the result of this is destructive to the whole peripteral system of flow, for the moment the pressure region commences to occupy the upper surface of the aerofoil a condition of instability arises and a new system of flow is inaugurated which produces a downward instead of an upward reaction. This is a fact easily demonstrated experimentally: a model in which the adjustment has been carried to its limits will behave in a most capricious manner, sometimes gliding perfectly and at others dropping suddenly in the midst of a flight like a bird when shot.

§ 189. The Hydrodynamic Standpoint.—Let us revert to the *Hydrodynamic* aspect of the subject as expounded in Chap. IV. We have seen that the supporting reaction is due to a cyclic motion in the fluid which is maintained by, and is in equilibrium with, the load on the aerofoil; it is of course understood that there is a superposed motion of translation, *i.e.*, the motion of flight. Now in the case of an aerofoil of infinite lateral breadth we have seen that this equilibrium is permanent, and we have in

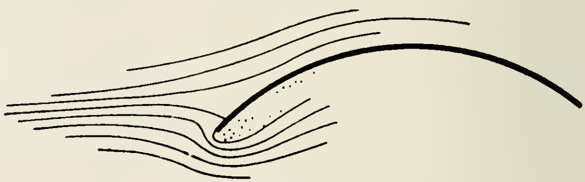


Fig. 118.

several instances plotted the resulting field of flow. When we have to deal with a case of finite lateral extent we have seen that there must be a continual *dissipation* of the cyclic motion, which vanishes in the manufacture of the trailing vortex filaments which the aerofoil is continually shedding on either hand.

In the Eulerian fluid there is no reaction on the aerofoil possible except that due to the cyclic motion, but in a real fluid this is not the case; a reaction may always be generated and always *is* generated when the motion gives rise to *discontinuity*, whether kinetic or physical.

Now the cyclic motion is in equilibrium with the reaction to which it gives rise, so that if we suppose an aerofoil supported *entirely* by the cyclic reaction and in equilibrium at any instant with the cyclic reaction, then if, firstly, it be supposed of infinite extent it will be in equilibrium at every other instant of time;

if, secondly, it be supposed of *finite* extent, then, since the cyclic motion is decaying, its equilibrium must vanish. In the case of the finite aerofoil we must consequently have the load in part supported by a reaction due to discontinuity, and in part only to the cyclic motion; the part of the reaction sustained by the discontinuity of motion may be regarded as that required to augment the cyclic motion at the same constant rate as its rate of decay. Hence: an aerofoil of finite lateral extent cannot be so designed that it shall be everywhere conformable to the lines of flow, and any such aerofoil must give rise to discontinuity in the motion of the fluid, involving surfaces of discontinuity, and presumably dead water regions.

We thus see that a perfectly conformable motion, such as we

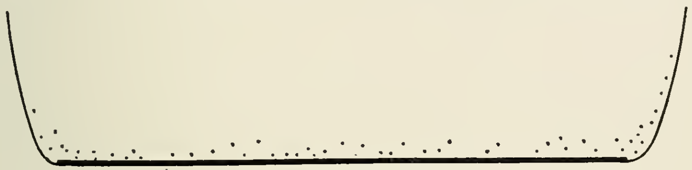


Fig. 119.

have tacitly supposed possible, is not possible when dealing with a real fluid, and at some point or points along its length the aerofoil must give rise to a discontinuity. This does not affect the validity of the foregoing theory, which has been founded on a hypothesis that admittedly does not fully represent the actual conditions; but it *may* be found that the matter now under discussion renders this hypothesis less valid than would otherwise be the case, especially where we are concerned with the quantitative estimation of the work done, *i.e.*, the computation of the gliding angle.

§ 190. Discontinuous Motion in the Periptery.—We may take as a simple example of the phenomenon under discussion the case of an aeroplane where, as we have seen, we have a system of flow of the Rayleigh-Kirchhoff type (Fig. 98). Let Fig. 119 represent such a plane in front elevation, then surfaces of discontinuity will spring from the ends in the manner shown, and these surfaces, constituting at first a Helmholtz vortex sheet, will break up into a number of vortex filaments which conceivably become the vortex cores discussed in Chap. IV. with reference to Figs. 83—86.

This view must at present be regarded as tentative, and is not altogether in agreement with the explanation put forward in the chapter to which reference has been made; it is highly suggestive, however, and on that account requires to be recorded; thus if we examine the wing of a bird we find that the middle portion is of a very characteristic arched section, but towards the extremities the arching is very much less pronounced; in fact, the form becomes such as might easily become the seat of discontinuity. This observation applies more particularly to the soaring birds, which in all respects constitute the best criterion.

If the view put forward in § 106 is correct, as to the cause of the noise made by bodies in rapid motion, then the "whirring" noise made by the wing of a bird in flight is a direct proof of the existence of discontinuity.

Of the two explanations of the genesis of the vortex continuations offered here and in Chap. IV., it is not necessary that either should be in error. The previous explanation also opens out a possibility that must not be lost sight of as bearing on the phenomenon now under discussion.

Let us suppose that the two air streams passing under and over the aerofoil find themselves when they meet at the trailing edge possessed of different velocities. Then their common surface would constitute a surface of discontinuity which might in itself fulfil the requirements of theory. But such a condition is impossible in an irrotational mass of fluid, for where there is a difference of velocity there is also a difference of pressure; and the fluid in the periptery is certainly irrotational in the sense of the argument.[1] But let us modify the supposition and take

[1] The continuity of the system of flow cut by a path taken round the aerofoil at some distance away, from one side to the other of the supposed

it that the two streams, although moving with the same velocity, are moving in different directions along the surface of separation; there would *appear* to be nothing contrary to hydrodynamic principles in this supposition; and the result would be a surface of discontinuity which *might conceivably* satisfy the condition.

The subject is one of very great difficulty, and it is impossible to do more than point out the more probable interpretations.

§ 191. Sectional Form.—The simple arc form of section employed as an illustration in § 188 is, qualitatively speaking, representative of that which may be considered essential, although the actual section more commonly resembles that given in Fig. 57, which may be regarded as typical.

We have seen that the consequence of an excessive "dip" on

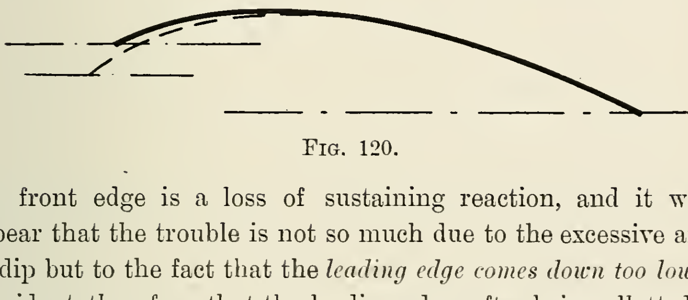


Fig. 120.

the front edge is a loss of sustaining reaction, and it would appear that the trouble is not so much due to the excessive angle of dip but to the fact that the *leading edge comes down too low*; it is evident, therefore, that the leading edge, after being allotted the theoretical angle and position, as in Fig. 120 (indicated by the dotted line), should be curtailed somewhat in the manner shown.

In the wings of birds the elastic nature of the trailing portion probably acts as a considerable safeguard, for should the pressure reaction show any sign of falling off, the elasticity of the plumage will immediately rectify matters; it is at least impossible to get any sudden reversal as may happen when the aerofoil is a rigid structure.

In § 120 it was suggested that the *form* of section might be

surface of discontinuity, points to this conclusion; the motion in the region traversed by such a path would be sensibly irrotational.

uniform throughout the length of the aerofoil, but of changing *scale*, *i.e.*, that it should tail off to a point at each extremity. Such a form is not generally found in Nature; the section nearly always becomes flatter and the angles of dip and trail become less as the extremities are approached. It is not known whether this fact is due to the reasons suggested in the preceding section, or whether it is attributable to aerodromic considerations alone, or whether there is some further subtle reason that has hitherto escaped detection. It is certain that the feature in question is valuable from the *aerodonetic* standpoint, and that is all that can be said with certainty at present.

§ 192. A Standard of Form.—In 1894 the author, with a view to embarking on some experiments in flight, took measurements of the plan-form of several of the soaring birds, including an albatros, a herring gull, and a kittiwake gull, with the result that an elliptical form was adopted as being a simple geometrical form whose ordinates approximate very closely to the average of those adopted by Nature. No attempt was made to imitate the "sinuosity" of the bird wing plan-form, this being regarded as an anatomical accident.

The form of section adopted has been given in Fig. 58; the aerofoil being made of timber, it was necessary to adopt a form easy to produce; on this account the hollow in the underside was very soon abandoned owing to the results not justifying the additional labour.

The "grading" is segmental or parabolic[1], that is to say, the maximum thickness of the section at different points along the length of the aerofoil is given by the ordinates of the segment of a circle whose cord is constituted by the flat face. This method of grading, independently of the plan-form to which it is applied, ensures the proper tailing off of the load towards the extremities as set forth in § 120. It is evident that if the thickness of the stratum at different points along the aerofoil is in constant

[1] For small amplitude the two curves do not sensibly differ.

relation to its width at each point, the mass dealt with per unit length will be everywhere as the width. But for a given height of the arched section the values of a and β will be inversely as the width, and consequently the load sustained for any particular element of the length will be constant in respect of width; that is to say, the sustaining power of the aerofoil for a given mid-section and a given *grading* is constant, no matter what the plan-form, both as to total and as to each element of length.

This is a very convenient rule to remember, but one which, from the nature of the assumptions made, is more or less

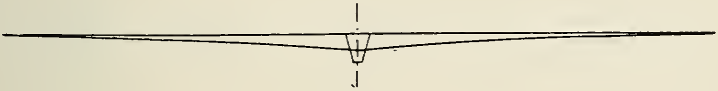


FIG. 121.

approximate; it can be applied legitimately to all ordinary modifications of plan-form.

When an aerofoil is designed according to the foregoing

FIG. 122.

specification, whether as a solid as in the case in point, or as a lamina of the same mean section, the *equivalent area* for uniform values of a and β will evidently be that of a plane whose plan ordinates are those of a segment, that is, proportional to the thickness ordinates. Such a form may be taken as having two-thirds the area of the circumscribed rectangle; that is, if L be the length of the aerofoil the equivalent area will be:—

$$\frac{2}{3} \times \frac{L^2}{n}.$$

By adopting and adhering to some standard such as that above defined, the experimental information obtainable becomes of greater value than when a variety of forms are employed. It is only desirable to modify the design when the data for some defined form is fully established and available as a standard of reference.

In experimenting with mica models the author has frequently adopted the form of natural flexion of an elliptical mica plate secured to a central bolster, to give the desired mid-section form; the *grading* is found to take the form given approximately in Fig. 121. In other cases the mica plate has been artificially graded to approximate to the standard given above, by fitting additional ribs as shown in Fig. 122.

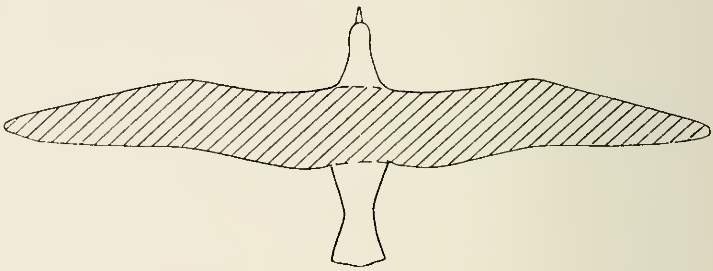


Fig. 123.

§ 193. **On the Measurement of "Sail Area."**—The appropriate measurement of the *sail area* or wing area of birds of various species is a rather vexed question. Some writers have regarded the wings as the sole organs of support, and the actual wing area alone has been reckoned as effective. Others (notably M. Moulliard) have assumed that the whole plan area (or *ombre*) of the bird contributes to the support, and have made pressure computations on this basis.

The author's view is that the influence of the body as a supporting member cannot be ignored, but that probably its effect can be best estimated as equivalent to an imaginary band of appropriate width forming a junction between the two extended wings as represented in Fig. 123. This view is based on the knowledge that the cyclic system must be continuous from wing to wing, and, on the whole, will produce a reaction on the body just as if the wing were continued in the manner shown; also on the improbability that there is any augmentation due to discontinuity, for the body is of a rounded form little likely to produce motion of a discontinuous type, and the tail is essentially a directive organ.[1]

We have seen that the different portions of a bird's wing are of different sectional form and adapted to carry different pressure values at different points. It sometimes happens that, for the purposes of comparison, in place of the actual area the *equivalent area* is required, as for example when comparing the results of theory, as in § 185, with the proportions adopted by Nature; this area being the area that would be required on an assumption of uniform pressure distribution, or constant values of α and β throughout the length.

In order to rightly assess this *equivalent area* we require to know the *grading* of the aerofoil, a knowledge which we do not possess, and which, owing to the flexibility of the wing structure, it is almost impossible to obtain.

The probability is that for birds of similar habits the grading will be found to be similar; that is to say, the distribution of the load along the length of the outstretched wings will be identical. In the absence of more definite knowledge, a rough assumption has been made for the purposes of the present work, *i.e.*, that the grading is substantially that of the *standard form* adopted by the author, and consequently the *effective area* is given by the expression $\frac{2}{3}\frac{L^2}{n}$.

§ 194. The Weight of the Aerofoil as influencing the Conditions of Least Resistance.—The subject of the influence of aerofoil weight as affecting the conditions of least resistance has been discussed in a previous chapter (§ 171), and a general equation has been deduced from the conditions.

[1] This point will be better appreciated when the aerodromic aspect of the subject has been discussed.

The most important application of the foregoing mathematical theory is found in the case where V is fixed by external conditions and where the area (A) is the variable; this application gives rise to pressure values greater than those given in Table X., the difference being dependent upon the extent to which aerofoil area is "penalised" by the direct effect of its weight in causing additional resistance.

We start the present continuation with Equation (3) of § 171:—

$$V^4 = \frac{a}{f\,L^4} + \frac{(2-q)\,b}{2\,f\,L^{4-q}} + \frac{(1-q)\,c}{f\,L^{4-2q}}. \qquad (3)$$

Substituting the constants in full we have—

$$V^4 = \frac{C_1\,W_1^2}{C_3\,\xi\,L^4} + \frac{(2-q)\,C_1\,W_1\,k}{C_3\,\xi\,L^{4-q}} + \frac{C_1\,k^2\,(1-q)}{C_3\,\xi\,L^{4-2q}},$$

whence

$$L^4 = \frac{C_1}{V^4\,C_3\,\xi}\,(W_1^2 + W_1\,k\,(2-q)\,L^q + k^2\,(1-q)\,L^{2q}). \qquad (6)$$

We now require to substitute for C_1 and C_3. These values were not investigated in § 171; they are obtained as follows:—

We know that—

$$W = \rho\,\kappa\,A\,V^2\,(\epsilon + 1)\,\beta$$

and

$$y = \frac{\rho\,\kappa\,A\,V^2\,(1-\epsilon^2)\,\beta^2}{2}$$

we require y in terms of W eliminating β, hence—

$$\frac{y}{W^2} = \frac{1-{}^2\epsilon}{2\,\rho\,\kappa\,A\,V^2\,(1+\epsilon)^2} = \frac{1-\epsilon}{2\,\rho\,\kappa\,A\,V^2\,(1+\epsilon)}$$

or

$$y = \frac{1-\epsilon}{2\,\rho\,\kappa\,(1+\epsilon)} \times \frac{W^2}{A\,V^2}$$

and if we define L as being $= \sqrt{A}$, C_1 is defined by expression

$$y = C_1\,\frac{W^2}{L^2\,V^2}. \qquad (\S\ 171.)$$

$$\therefore \qquad C_1 = \frac{1-\epsilon}{2\,\rho\,\kappa\,(1+\epsilon)}.$$

Now C_3 (see § 171) is defined as such that the quantity $C_3\,\xi\,V^2\,L^2$ is the skin friction on the aerofoil, but we know that this is given in full by the expression—$\epsilon\ C\ \rho\ V^2\ A$ and $L^2 = A$, therefore $C_3 = C\ \rho$.

Substituting for C_1 and C_3 in Equation (6) we have—

$$L^4 = \frac{1 - \epsilon}{V^4 \times 2\ \rho^2\ \kappa\ (1 + \epsilon)\ \xi\ C} \times (W_1^2 + k\ W_1\ (2 - q)\ L^q + k^2\ (1 - q)\ L^{2q}).$$

§ **195. A Numerical Example.**—The employment of this equation may be illustrated by a numerical example. Let us take the case of an aerodrome of 1,000 poundals *essential* weight (31 lbs. approximately), to be designed for a velocity of 50 feet per second. The remaining data are as follows:—

n taken as 12

therefore— $C = \cdot 75$

$\epsilon = \cdot 75$

$\kappa = 1{\cdot}2.$

It is found by trial design or by calculation that in the expression $W_2 = k\ L^q$ that—

$k = 50$

and $q = 1{\cdot}5.$

We take $\xi = \cdot 03$

and $\rho = \cdot 078.$

Substituting values in Equation (7) we obtain —

$$L^4 = \frac{\cdot 25}{6{,}250{,}000 \times 2{\cdot}4 \times \cdot 0061 \times 1{\cdot}75 \times \cdot 03 \times \cdot 75} \times (1{,}000{,}000 + 25{,}000\ L^{1\cdot 5} - 1{,}250\ L^3)$$

$$= 70 + 1{\cdot}75\ L^{1\cdot 5} - \cdot 0875\ L^3 \text{ (approx.)}.$$

This *form* of equation can only be solved by plotting or by guessing[1]; the solution gives $L = 2{\cdot}96$, that is to say, the area $A\ (= L^2)$ is 8·76.

Now the total weight sustained by this area is $W_1 + W_2$ and $W_2 = 50\ L^{1\cdot 5} = 254$ (poundals), or $W_1 + W_2 = 1{,}254$, or 39 lbs. almost exactly. We therefore have pressure per square foot

[1] The particular case of $q = 1{\cdot}5$ is an exception.

for least resistance $= \frac{39}{8\cdot76} = 4\cdot45$ as against $3\cdot70$ as given in Table X.

We can calculate the corresponding value of β from the expression $\frac{P_2}{V^2} = \rho\,\kappa\,(\epsilon + 1)\,\beta$ (§ 185), or $\beta = \frac{P_2}{\rho\,\kappa\,(\epsilon + 1)\,V^2}$ which in the present example gives $\beta = \frac{\cdot0585}{\rho\,\kappa\,(\epsilon + 1)} = \frac{\cdot0585}{\cdot164} = \cdot356$, or $20\cdot4$ degrees against $16\cdot8$ degrees according to Table IV.

It would thus appear that unless the aerofoil weight in the above example has been greatly exaggerated, its influence on the conditions of least resistance is a fact that should certainly be taken into account. At present the accuracy with which the fundamental data have been ascertained would scarcely justify the preparation of Tables to include the influence of this factor.

§ 196. The Relative Importance of Aerofoil Weight.—The importance of the present branch of the subject evidently becomes greater with any increase in the size of the aerodrome, for the necessary proportionate weight of the aerofoil will be greater on a large aerodrome than on a small one; this fact is almost self-evident, but is in any case easy of proof.

Let the weight of the aerofoil, as in the preceding section, be represented by W_2, and as before let W_1 be the essential load; then we have seen that we can represent W_2 by the approximate expression— $W_2 = k\,L^q$. We assume that the weight of the aerofoil itself does not materially add to its stresses, being supported *directly*.

Now let us suppose, as is the case for *similar* bodies, that W_1 varies as L^3, then W_2 will vary as $W_1\,k\,L^q$, that is, as $k\,L^{3+q}$, or the relation of W_2/W_1 will be represented by $\frac{L^{3+q}}{L^3} \times$ constant, that is, $L^q \times$ constant. Now, if the index q were as low as 1 (and it is improbable that it is lower) the relative weight of the aerofoil W_2/W_1 will increase as the linear dimension. If q be greater the increase will be appropriately more rapid.

The relation W_2/W_1 could only remain constant if the value of q were to sink to zero, a condition which is manifestly impossible.

It is probable that for an aerodrome the size of that chosen as an example in the preceding section the value of W_2/W_1, is in excess of that necessary, but it is questionable whether it would be possible to construct a machine of moderate velocity capable of bearing a man, without the aerofoil considerably exceeding one-fifth of the total weight.

The question of aerofoil weight as affecting the phenomenon of flight as presented by Nature takes its place in the later portion of the work. It is only necessary here to point out that the larger birds must, in general, be more influenced than the smaller ones, just as in the case of other forms of *aerodrome*; and in the matter under discussion we have one of the causes, if not the most important cause, that constitute determining factors of that critical point at which Nature finds it advantageous to change from the insect mode to the avian mode of flight: that is, from the aeroplane to the pterygoid form.

NOTE.—Tables, etc., in present chapter relate strictly to incompressible fluid. For method of taking compressibility of air into account, see Appendix I.

CHAPTER IX.

ON PROPULSION, THE SCREW PROPELLER, AND THE POWER EXPENDED IN FLIGHT.

§ 197. Introductory.—The employment of the Newtonian method (§ 2) in the theory of propulsion has been already mentioned (§ 8). The application of this method, which constitutes the foundation of modern theory, owes its development principally to the work of Rankine and W. Froude.

In the *general theory of propulsion* we are not concerned with the machinery of propulsion, *i.e.*, the form of propeller—paddle, screw, jet, or other known or unknown mechanism; we merely take account of the fact that forces are exerted between the propelled body and certain parts of the fluid, and investigate the conditions that obtain and the proportion of power that may be utilised and lost. The theory of propulsion on this broad basis is the common foundation of propeller theory generally, and the conditions deduced from the Newtonian principle are essential to every form of propeller. It is convenient, in the initial consideration of the problem, to introduce the notion of action at a distance, and to suppose the propulsive forces to consist of repulsions (or attractions), acting in the direction of motion between the propelled body and the particles of the fluid.

§ 198. The Newtonian Method as applied by Rankine and Froude.—It is supposed in the first instance that the fluid on which the propeller operates is at *rest* at the time the propulsive forces commence to act; this condition is intended to exclude any possible disturbances that may be set up in the fluid by the body in motion.

Let **F** be the sum of the propulsive forces.

„ **m** „ mass of fluid handled per second.

„ V „ velocity of vessel, the term *vessel* being used to denote the body propelled.

„ **v** „ a uniform sternward velocity imparted to the fluid operated upon.

All in Absolute Units.

Then we have (§ 3) $\mathbf{F} = \mathbf{m}\,\mathbf{v}$, and the work done *usefully* per second is

$$\mathbf{F}\,V = \mathbf{m}\,\mathbf{v}\,V \qquad (1)$$

and the energy left in the fluid per second, that is, *lost power*, is

$$= \frac{\mathbf{m}\,\mathbf{v}^2}{2} \qquad (2)$$

or total energy per second

$$= \mathbf{m}\,\mathbf{v}\,V + \frac{\mathbf{m}\,\mathbf{v}^2}{2}$$

or *efficiency*

$$= \frac{\text{Useful work}}{\text{Total work}} = \frac{V}{V + \frac{\mathbf{v}}{2}}. \qquad (3)$$

This, according to Rankine,[1] is *the theoretical limit to the efficiency of a propeller.* It will be shown subsequently that this assertion requires qualification.

If we depart from the simplicity of the assumption and suppose that the different portions of the fluid acted upon receive different velocities, the foregoing demonstration requires appropriate modification; the **v** of expression (1) and the **v** of expression (2) are not the same quantity, the $\mathbf{v}^2$ in the latter expression becoming the *mean square* of the velocity **v** instead of the *square of the mean.* For a given value of **m** the efficiency must thus be less than if the velocity **v** were uniform over the

[1] "Miscellaneous Scientific Papers," Rankine, XXXIII.

mass, for the *mean square* is always greater than the *square of the mean.*

The full expression where the velocity is variable throughout the mass m is,

$$\frac{V}{V + \frac{\mathrm{v}}{2} + \frac{\Sigma\, \mathrm{m}\, (x^2)}{2\, \mathrm{m}\, \mathrm{v}}}, \qquad (4)$$

where x represents the velocity communicated to the particles of fluid in excess or in deficit of the mean, x being accordingly *plus* or *minus*. It will be noted that since x^2 must be always positive the quantity $\frac{\Sigma\, \mathrm{m}\, (x^2)}{2\, \mathrm{m}\, \mathrm{v}}$ will be always positive, so that the efficiency will be less than if the mass were handled uniformly. The above expression is of but slight utility from a quantitative standpoint; it is given here as being conducive to exact thought and as being the more complete form of expression (3).

§ 199. Propulsion in its Relation to the Body Propelled.—In the preceding section the subject of propulsion has been treated in the abstract; it has been assumed that the body propelled is far away so that the fluid is unaffected by its presence, and that the fluid as a whole receives momentum.

Now we know from *the Principle of No Momentum* (§ 6) that, as a whole, the fluid does not receive momentum, and that if it receive momentum in one direction in one part it simultaneously receives equal and opposite momentum in some other part. The result of this is two-fold: (*a*) we know that the whole of the energy expended in the fluid does not appear as sternward motion, as assumed by Rankine; and (*b*), the problem becomes complicated by the reaction and motions produced in the fluid by the vessel itself as affecting the conditions under which the propeller is working.

For reasons stated in § 8 it is doubtful whether, under the conditions that ordinarily obtain, the error that arises from

(a), neglecting the counter-current, is sensible; if, however, the proportion v/V were to become considerable, the departure from theory would become serious. The validity of the present application of the Newtonian method depends definitely upon the fact that v/V is small.

Considering next (b) the problem of propulsion in its full relation to the body propelled: if a vessel be *towed* through a fluid, the pull on the tow line being applied *from without*, the whole energy is, in the sense of § 198, usefully employed, and the condition of affairs is that tacitly taken in § 198 to represent unit efficiency.

The energy expended passes into the fluid and is swallowed up partly in overcoming viscous stress and partly in setting the fluid in motion; the first part vanishes at once into the thermodynamic system and is lost; the second part remains in the fluid as kinetic energy until in turn it is spent in overcoming viscosity, when by degrees it also vanishes.

Now the energy that is left in the kinetic form takes some time to disappear, and in the meantime it constitutes a wake current with a corresponding *counterwake* whose momenta are equal and opposite, the wake current being situated in the immediate track of the vessel, and the counterwake further afield and extending theoretically to the confines of the fluid region, but only sensible for a limited distance. This kinetic energy is not irretrievably lost; after the fluid has passed *in effect* out of dynamic connection with the vessel, there is ample time for the recovery of the energy if a suitable method could be devised, and then, by returning it to the source of power, it is evident that the vessel will be propelled with a less expenditure of power than previously, and the "*efficiency*" will become greater than that somewhat arbitrarily chosen as unity.

§ 200. A Hypothetical Study in Propulsion.—Let us consider the case in which the propeller is *constrained to act on the wake current* (or, as it is sometimes termed, "the frictional wake"), and in the first instance let us suppose that this current consists of a quantity of fluid moving *en masse* with a velocity $= \mathbf{v}_1$. Now the force of propulsion is essentially equal, and of opposite sign, to the resistance experienced by the vessel, action and reaction being equal and opposite; consequently, on the Newtonian basis, the rearward momentum communicated by the propeller will be equal to the forward momentum communicated by the vessel, so that the conditions of propulsion will be satisfied if the propeller impart to the wake current a rearward velocity $\mathbf{v}$ equal to $\mathbf{v}_1$; that is to say, the fluid will be brought to rest.

Let us now re-calculate the efficiency as in § 198: we have work done usefully per second $= \mathbf{F}\, V = \mathbf{m}\,\mathbf{v}\, V$; and energy *taken out of the fluid*, that is, energy *received* per second, is $= \frac{\mathbf{m}\,\mathbf{v}_1^2}{2}$; or,

total energy expended per second $= \mathbf{m}\,\mathbf{v}_1\, V - \frac{\mathbf{m}\,\mathbf{v}_1^2}{2}$, or,

$$\textit{efficiency} = \frac{V}{V - \frac{\mathbf{v}_1}{2}}. \qquad (5)$$

This is *greater than unity;* the result being, as anticipated, that it is *theoretically possible* that a vessel should be *propelled* for a less expenditure of power than that by which it can be *towed.*

This important result, although not generally known,[1] is not new; it was previously pointed out by Mr. W. Froude in the discussion on a paper by Sir F. C. Knowles (*Proc. Inst. C. E.* 1871). Froude evidently had also treated the matter quantitatively, since he mentions the theoretical possibility of a *negative slip* of a screw propeller, from the cause stated, equal to half the positive slip as ordinarily computed.

In the present hypothetical case the influence of the counterwake has not been taken into account, it forming no part of the Newtonian scheme; the conditions are too artificial for the omission to be a matter of any importance, apart from the fact

[1] The author has known this result received with *open incredulity* by persons considered to be authorities on propulsion.

already pointed out that the consequences of such an omission will not be serious.

It has been assumed that the whole resistance of the vessel is due to its skin friction. In marine propulsion "wave making" plays a prominent part; the resistance from this cause may be regarded as a force *applied from without*, since the waves travel away, taking their momentum with them; the consideration of wave making resistance would destroy the precise balance between v and v_1, on which expression (5) is based; the matter of wave making has, however, no interest to us from an aerodynamic standpoint.

§ 201. Propulsion under Actual Conditions.—Under actual conditions neither of the hypotheses discussed in §§ 198 and 200 applies in its entirety. The requirements of the former hypothesis are most nearly met in the case of a paddle boat (with the paddles at the sides); the latter case is best exemplified in the stern-wheeler, a flat-bottomed type of craft whose propeller is particularly well placed for capturing the frictional wake. In the forms that succeed in practice the propeller is *usually* behind in the frictional wake, *never* in front; and the successful forms of propulsion are those in which a sufficient mass of water per second can be conveniently handled; thus, *jet* propulsion has become practically extinct. We are, therefore, led to appreciate the soundness of the Newtonian method.

There are many methods of mechanical propulsion, that is to say, there are several known mechanical devices for producing the reaction on the fluid which we have so far regarded as being accomplished by *action at a distance*. Firstly, we have the numerous devices employed by nature in the locomotive mechanism of birds, fishes, etc.; secondly, we have the primitive devices employed by man—the paddle, the oar, etc.; and finally, we have the two great inventions of marine engineering, the paddle wheel and the screw propeller. Of these various types of propeller, only two will be discussed as of interest in connection with the subject of the present work, the *screw propeller* and *wing propulsion*; the former alone being deemed suitable for treatment in the present volume, the latter being reserved for the section on "Avian Flight" which will form part of Vol. II.

§ 202. **The Screw Propeller.**—We will presume a general knowledge of the screw propeller, and proceed at once to the attack.

The theory of the screw propeller will be discussed on the basis of the peripteral theory of the foregoing chapters; this constitutes a new method which sheds considerable light on a hitherto somewhat obscure subject.

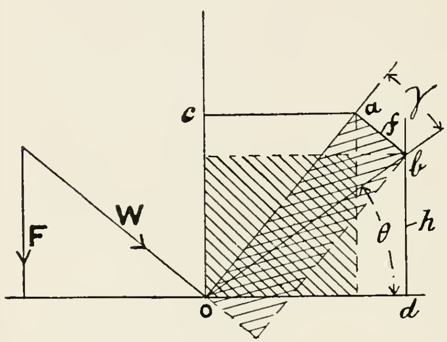


Fig. 124.

We shall in the following demonstration take the helical surface of uniform pitch as strictly the analogue of a plane in the foregoing theory, and we shall presume that the various propositions already proved in the case of the aerofoil apply *mutatis mutandis* to the helical equivalent. Thus the blade of a propeller becomes an aerofoil of a form suitable to glide in a helical path, the reaction on the blade (whose resolution is the torque and thrust) is the analogue of the *weight*, the helical surface at right angles to the blade reaction is the analogue of the horizontal plane, and concentric cylindrical surfaces represent vertical planes in the axis of flight.

We will begin by an examination of an element of a propeller blade represented by its section on one of the aforesaid cylindrical surfaces, of which we will suppose the development is given in Fig. 124. Now, on this development a helical surface will appear as an inclined straight line; let $O\,a$ represent the helical surface which we regard as the analogue of the horizontal, and **W** (the analogue of W) the force at right angles thereto. Let $O\,b$ be the helical flight path, and γ the gliding angle; then θ, the angle cut off between $O\,b$ and the axis of x, will be the effective pitch angle; that is to say, the line $O\,b$ represents the helix along which the blade of the propeller will actually travel, and its pitch will be the *effective* pitch of the propeller.

Draw the line **F** parallel to the axis of y to represent the component of **W** in the direction of motion of the vessel, cut off $O\,a$ = **W**, and draw $a\,c$ perpendicular to $O\,y$, draw $a\,b$ perpendicular to $O\,a$, and $b\,d$ perpendicular to $O\,x$.

Then $O\,a$ being equal to **W**, we have $a\,c$ equal to **F**, the two triangles being equal in every respect. Let us denote $a\,b = f$, and $b\,d = h$.

Now while the blade moves from O to b the *energy lost* will be $= \mathbf{W}\,f$; that is to say, we regard the matter as a case of *gliding*, to which it is strictly analogous. The energy utilised in propulsion during the same period will be $= \mathbf{F}\,h$. (These quantities are indicated by the shaded areas in the figure.)

Now it follows from the construction that—

$$\mathbf{F} = \mathbf{W}\cos(\theta + \gamma),$$

$$f = \mathbf{W}\tan\gamma,$$

$$h = \mathbf{W}\frac{\sin\theta}{\cos\gamma}$$

$$\therefore \quad \frac{\mathbf{W}\,f}{\mathbf{F}\,h} = \frac{\tan\gamma\cos\gamma}{\cos(\theta+\gamma)\sin\theta}$$

or,

$$\frac{\textit{Energy lost}}{\textit{Energy utilised}} = \frac{\sin\gamma}{(\cos\theta\cos\gamma - \sin\theta\sin\gamma)\sin\theta}$$

$$= \frac{1}{\cot\gamma\cos\theta\sin\theta - \sin^2\theta}. \qquad (1)$$

§ 203. Conditions of Maximum Efficiency.—The conditions of maximum efficiency are attained for the element of the propeller under consideration, when (1) is solved for minimum value.

Firstly, we may note from Equation (1) (and it is otherwise self-evident) that γ should be made as small as possible; that is to say, the propeller blade should be designed as an aerofoil for minimum gliding; we shall therefore take γ from this point to denote the *minimum* gliding angle as independently ascertained. If the conditions of *least gliding* angle are infringed, the present theory continues to apply, but the result will be the best efficiency possible under the adverse conditions imposed, and not the real maximum.

Now γ, and therefore $\cot\ \gamma$, is constant in our expression; consequently we have to solve—

$$\cot \gamma \sin \theta \cos \theta - \sin^2 \theta$$

for maximum value where θ is the only variable. Differentiating in respect of θ we get—

$$\cot \gamma \cos 2\theta - 2 \sin \theta \cos \theta = 0$$

or

$$\cos 2\theta = 2 \sin \theta \cos \theta \tan \gamma$$

transforming we get—

$$\cot 2\theta = \tan \gamma,$$

hence

$$\theta = \frac{90^\circ - \gamma}{2}$$

which we may express in another way and say: *The angle of greatest efficiency is* 45 *degrees*, minus *half the least gliding angle.*[1]

Thus, if we take the gliding angle in the case of *air* to be 10 degrees for any particular aspect proportions, the angle of greatest efficiency will be 40 degrees; or, taking the probable equivalent for water as 6 degrees, the appropriate angle becomes 42 degrees. The figures cited are *probable* figures for blades of about 4 : 1 ratio, *as founded on experiment;* it is known that the tabulated theoretical figures of § 181 are too low, from causes already discussed.

§ 204. Efficiency of the Screw Propeller, General Solution.—From Equation (1), § 202, we obtain

[1] It is worthy of remark that the solution is the same as for a *solid screw;* the constancy of the gliding angle renders it analogous to the angle of friction.

$$Efficiency = 1 - \frac{1}{\cot\gamma\ \cos\theta \sin\theta - \sin^2\theta + 1}$$

From which by transformation,

$$Efficiency = \frac{\tan\theta}{\tan(\theta + \gamma)}.$$

This expression may be deduced directly from the conditions. Let Fig. 125 represent, by the lines *O a* and *O b*, the helices of horizontal and gliding path respectively; then, since the reaction **W** is normal to the path *O a*, the *work done* when the propeller is rotated through an angle represented by the line *O d* will be **F** × *a d;* but the work utilised is represented by **F** × *b d;* the efficiency is therefore $\frac{b\ d}{a\ d}$, that is, $\frac{\tan\theta}{\tan(\theta + \gamma)}$.

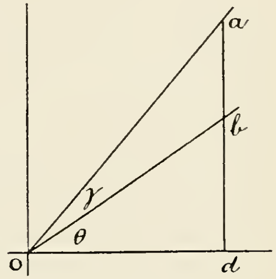


Fig. 125.

The result is thus obtained in a more direct manner, all trigonometrical transformations being dispensed with; the original demonstration is, however, of a more exact nature and is based on a clearer conception of the conditions involved. The identity of the two methods may be demonstrated geometrically by showing that the shaded areas of Fig. 124 are proportional to the ordinates *b d* and *b a* of Fig. 125, a matter of perfect simplicity.

The present theory enables us to define the *slip* of the propeller as the difference between the ordinates *b d* and *a d*, the slip ratio being represented by $\frac{a\ b}{a\ d}$. The term slip as here defined is not identical with the slip of the naval architect, which is derived from the *mean pitch angle of the blades*, a basis that can have no justification in theory. The conception of *slip* originated with the propeller of true helical form and then denoted the difference between the geometrical and effective pitch; when blades were given an *increasing pitch* the mean pitch angle was taken as the basis of calculation of the geometrical pitch; hence the present usage. The term *slip*, in its application to a screw propeller, is one that leads to confusion of thought; it is unscientific in its present usage, and would be better abolished.

Let us estimate the possible efficiency in the case of the maximum conditions of the preceding section. Employing equation, E (efficiency) $= \dfrac{\tan\theta}{\tan(\theta - \gamma)}$ we have —

Air $(\theta = 40°)$ $E = 70{\cdot}4$ per cent.
Water $(\theta = 42°)$ $E = 81{\cdot}1$ per cent.

Now these efficiencies are only obtainable at the particular *section* of the blade where the angle θ is correct for maximum efficiency, and at all other points the efficiency must be less. A section of the blade here discussed is the section on a cylindrical surface which may be fully defined by its radius r.

In Figs. 126 and 127 we have plotted the calculated efficiency for different values of r on the basis of the γ values assumed for water and air. Abscissae represent radius in terms of pitch, the ordinates give corresponding efficiency values. The abscissae are also figured for values of θ. The efficiency falls to zero when $\theta = 0$, and again when $\theta + \gamma = 90$ degrees, for in the first place $\tan\theta = 0$, and in the second $\tan(\theta + \gamma)$ becomes infinite.

§ 205. The Propeller Blade Considered as the Sum of its Elements.— Much of the faulty work of early writers on hydrodynamic problems has been due to the treatment of a body or surface as the equivalent of the sum of elements into which it may be arbitrarily divided, and this form of error is one against which it is important to be on guard.

We have already adopted in substance the principle of regarding an aerofoil as the sum of its sectional elements in the sense now contemplated in respect to the propeller blade (§ 192), but we do not suppose, in assessing the individual elements, that they are *removed from their environment.*[1]

[1] A frequent cause of error in the work of writers of some forty or fifty years ago.

Let us examine our former procedure. We have an aerofoil whose aspect ratio is of considerable magnitude, and whose *grading* is specified, and we prove that the reaction due to each increment of length is proportional to the *grading ordinate* proper to that increment, irrespectively of the fore-and-aft dimension. The proof involves the tacit assumption that for the smooth curve form of grading specified, a *geometrical similarity* of section at all points involves a uniform pressure distribution.

Now, so long as the aspect ratio is great and the grading a smooth curve, there can be little question as to the propriety of this assumption; if, however, the aspect ratio be small, or the changes in the grading curve sudden, then the grading curve and the relative reaction curve may cease to coincide. Our assumption may therefore be considered as an approximation—a close approximation when the value of n is great (say upwards of 4 or 5), and a rough approximation when the aspect ratio is small.

§ 206. Efficiency Computed over the Whole Blade.—On the basis of the efficiency curve (Figs. 126 and 127) and a knowledge of the radial distribution of the thrust reaction (the **F** of § 202), the computation of the efficiency for the whole blade is merely a matter of integration.

We have first to settle how much of the efficiency curve we propose to employ, *i.e.*, the radial limits of the blade length. If we make the blades too long, the efficiency at the extremities would be so low as to involve an extravagant expenditure of power; if, on the other hand, we confine the length of the blade to the region where the efficiency is about its maximum, in order to reap the benefit of the full value (as given in § 204), we encounter practical disadvantages in the increased propeller diameter required to deal with a given quantity of fluid (the proportion of the "disc" area utilised becoming small), and in the length (and consequent resistance) of the arm necessary to attach the blade to its boss.

In Figs. 126 and 127 we have taken the blade length,

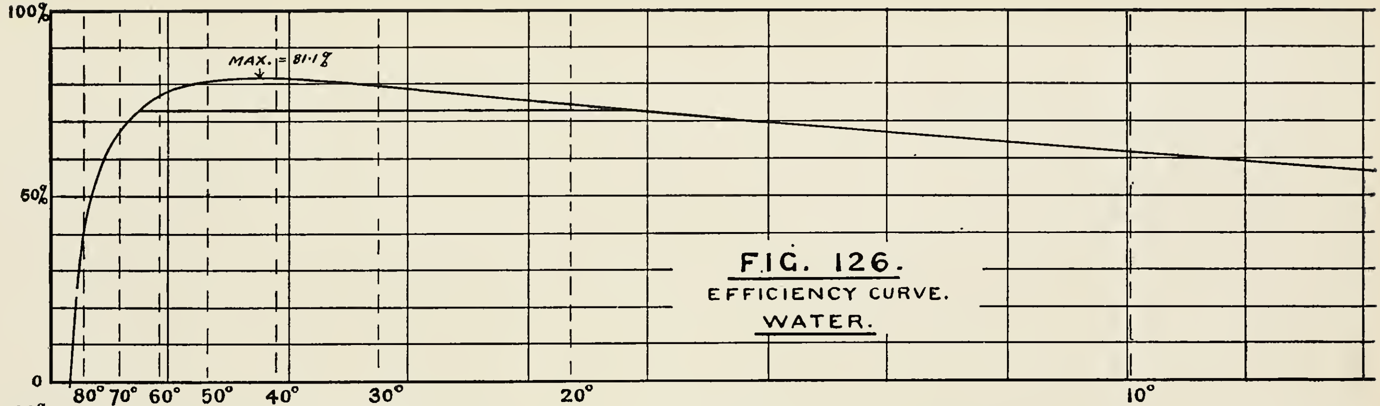


FIG. 126.
EFFICIENCY CURVE.
WATER.

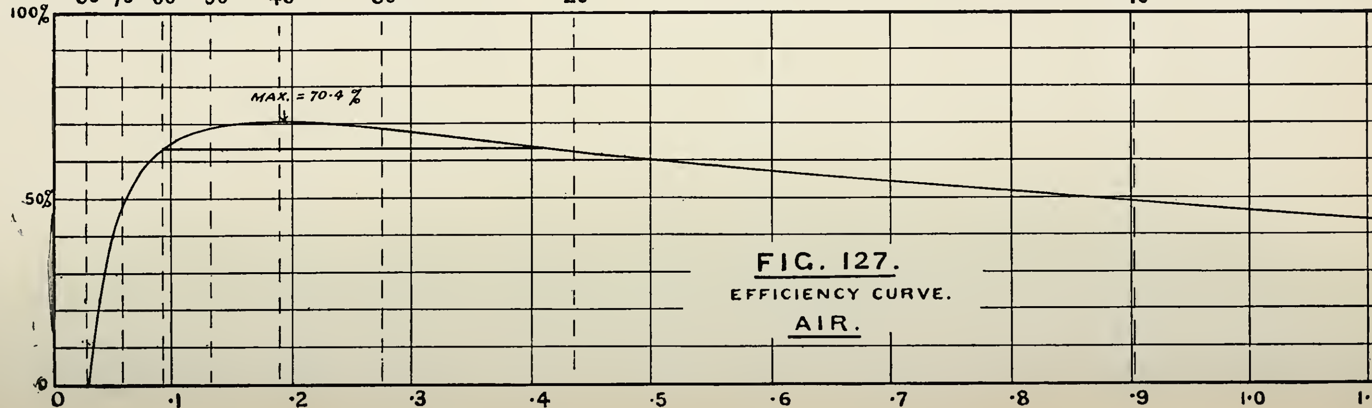


FIG. 127.
EFFICIENCY CURVE.
AIR.

indicated by the horizontal line, as confined to a zone in which the inferior limit of the efficiency is 90 per cent. of its maximum. It may be noted that this procedure gives for water a pitch almost exactly equal to the diameter; this is somewhat less than is customary, the generally accepted proportion being, pitch $= 1\frac{1}{4}$ diameter (approximately). It would appear that designers unconsciously reject the whole of the curve where of less value than about 95 per cent. of the maximum possible. In spite of this fact, the efficiencies so far recorded leave much to be desired, and, apart from practical limitations, would appear to show that there is still considerable room for improvement. The defects in existing practice would seem, according to the present theory, to be found in the want of attention to the requirements of pterygoid section, in the low value of **n** (aspect ratio) commonly adopted (possibly from practical requirements not included in the present hypothesis), and to undue fulness of plan-form towards the blade tips,[1] and consequent excessive frictional loss.

The value of the total efficiency, having selected the blade limits, depends not only upon the efficiency curve, but also on the distribution of the thrust over the length of the blade, or the *thrust grading*, as it may be conveniently termed. If the range of efficiency were very great, we should have to specify the thrust grading before the total efficiency could be computed; but as the variation does not usually exceed 10 per cent. or so, and as the general character of the grading curve cannot be in doubt, we can arrive immediately at a close approximation.

On the 90 per cent. basis, if the thrust grading were *uniform* over the length of the blade, the mean efficiency would, for the character of the curve given (Figs. 126, 127), be 96 or 97 per cent. of its maximum. If, as must be the case, the thrust grading fall to zero at the extremities, the efficiency will be increased;

[1] Compare §§ 214-216.

hence we may take it as probably about $97\frac{1}{2}$ per cent. of its maximum. On the basis of § 204 this gives

For air . . . . . . 68·6 per cent.
For water . . . . . . 79·0 per cent.

We have no data of comparison in the case of air; in the case of water, so far as the author is informed, the highest actual efficiency recorded is somewhat over 70 per cent.

§ 207. **Pressure Distribution.**—It is evident that, according to the present theory, the propeller blade is amenable to precisely the same laws so far as its pressure-velocity relation is concerned as the ordinary aerofoil, and we presumably also have the two alternative types of fluid motion, the continuous and the discontinuous, according as the blade is given a pterygoid form (based on a helix) or whether a simple helical surface or sheet (the analogue of an aeroplane) is employed. We may read the appropriate pressure for air from either Table X. or XII., as the case may be.

A complication is introduced in the propeller blade by the fact that its different portions are moving at different velocities through the fluid, so that the pressures proper to least γ vary at the different points along the length of the blade. This velocity, the V of the propeller blade, will be given by the expression $V = \frac{\mathbf{V}}{\sin\theta}$ where $\mathbf{V}$ is the velocity of the vessel, or, in terms of r, we have

$$V = \frac{\sqrt{(2\,\pi\,r)^2 + p^2}}{p}\mathbf{V},$$

where p is the pitch, or

$$V^2 = (4\,\pi^2\,r^2 + p^2)\,\frac{\mathbf{V}^2}{p^2},$$

and since p and $\mathbf{V}$ are constants the curve is of the form $y = x^2 +$ const., when plotted (Fig. 128), where abscissae give radius in terms of pitch and ordinates give V^2 values. Now by § 185, for any aerofoil $P_2 = V^2 \times$ const., values of the constant being given in Table IX. Hence the curve (Fig. 128) may be taken to give the correct pressure value for all points along the blade, the *pressure scale* being determined by assigning a value to some convenient point from the Table.

§ 208. **"Load Grading."**—Reference has already been made to the term *thrust grading* as representing the distribution of the sternward reaction along the length of the blade.[1] We have now to discuss the considerations governing the form of the *thrust grading curve*, and also the curve of distribution of the normal

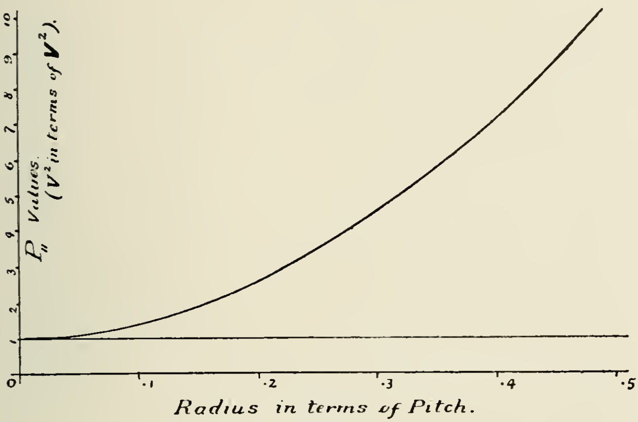


FIG. 128.

reaction from point to point, which we may term the *load grading*.

Dividing the propeller disc into a number of concentric areas, we have, on the principle discussed in § 198, to distribute the momentum as nearly as may be possible in proportion to the mass of fluid passing through each annular element: that is, the force exerted by each small linear element of the blade should be proportional to the area it sweeps; that is to say, it will be proportional to r.

Our thrust grading curve on this basis would be that shown in Fig. 129 (a)—simply an inclined line. But we must complete

[1] § 206.

this curve at the blade extremities. This has been done arbitrarily by a pair of ordinates, the thrust grading curve thus completed being the contour of the shaded area, the *area* itself representing the total thrust. But this "curve" infringes a condition already laid down, that the grading must constitute a smooth curve with no sudden changes of ordinate. Hence we must compromise, and we find that the combined conditions indicate a form such as that illustrated by Fig. 129 (*b*).

Now the reaction normal to the blade will at every point be equal to the sternward component multiplied by sec $(\theta + \gamma)$, that is to say, if we multiply the ordinates of the *thrust grading curve* at every point by sec $(\theta + \gamma)$ we shall have the *load grading*

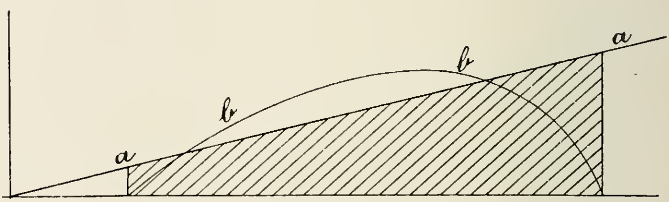


FIG. 129.

curve Fig. 130 (compare Fig. 136), which represents the distribution of the pressure normal reaction along the length of the blade.

§ 209. Linear Grading and Blade Plan Form.—The linear grading for any radius is the quotient when the load value is divided by the pressure value for that radius; thus the linear grading curve may be plotted from the other two, the ordinates being calculated by simple division (Fig. 130).

This linear grading is analogous to the aerofoil *grading* of § 192, and likewise represents the ordinates of the blade plan, *i.e.*, the width of the blade from point to point for constant *form* of section; that is to say, all sections become geometrically similar.

Whether or not the similarity of sectional form is essential, as it would theoretically appear to be for best economy, must be regarded to a certain extent as an open question. The same query has arisen in the case of the aerofoil, but the objections in that case are partly concerned with aerodromic and other considerations. Should future experience show that flattening of the section (compare § 191) and diminution of pressure

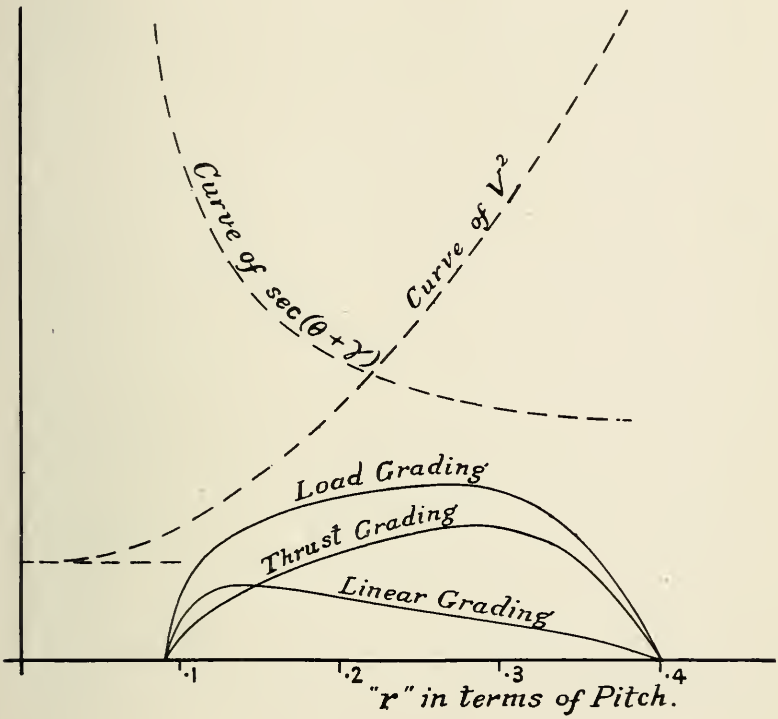


FIG. 130.

towards the extremities is advantageous from the aerodynamic point of view, the whole matter will require to be thoroughly reinvestigated before we can regard the theory of peripteral motion as complete.

§ 210. The Peripteral Zone.—Before we are in a position to discuss the conditions that regulate the *number of blades* permissible in a propeller we require to somewhat extend our knowledge of the dynamics of the periptery.

At first sight it might be supposed that the blades of a propeller in their helical paths are related in a similar manner to a number of superposed aeroplanes, and the law of maximum proximity will be the same in both cases. Such is not the case. Where we have to deal with a battery of superposed planes or aerofoils, whether vertically over one another, as in Fig. 131 (*a*), or (in order to better simulate the conditions) like a flight of steps (Fig. 131 (*b*)), the supporting reaction is continually derived from the virgin fluid, and the line of pressure reaction of any plane, or any component of it, *only cuts the path of that plane once.*

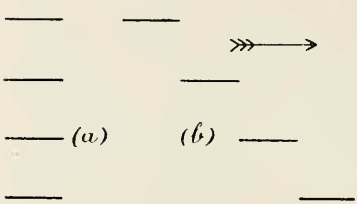


Fig. 131.

In the case of the propeller, on the contrary, the component of the pressure reaction of any blade in the line of motion cuts the paths of that blade *an indefinite number of times.* We have here to deal with a fact new to our theory.

Let us suppose that we substitute for our propeller blade some device that acts directly on the fluid without involving the complexity of the cyclic or *peripteral* motion, and let us stipulate that this hypothetical device produce the *same total reaction* with the same expenditure of energy as the original aerofoil or propeller blade.

On the Newtonian basis we know (§ 3) that if W be the total reaction, m the mass *per second* of the fluid dealt with, and v be the velocity imparted in the direction of the reaction—

$$W = m\,v \quad \text{or} \quad v = \frac{W}{m}.$$

But the energy per second $= \dfrac{m\,v^2}{2}$

$\therefore$ Energy per second $= \dfrac{W^2}{2\,m}$ or if W is constant, *energy per second* $\propto \dfrac{1}{m}$ (1)

Now, energy per second for pterygoid aerofoil handling mass per second $= m_1$ is

$$\frac{m_1 V^2 (\beta^2 - \alpha^2)}{2},$$

and energy per second on the simple Newtonian basis for the *same mass* handled per second would be

$$\frac{m_1 V^2 (\beta + \alpha)^2}{2},$$

or, by (1), if m_2 is the mass per second dealt with by our hypothetical device, we shall have

$$\frac{m_2}{m_1} = \frac{m_1 V^2 (\beta + \alpha)^2}{m_1 V^2 (\beta^2 - \alpha^2)} = \frac{(\beta + \alpha)(\beta + \alpha)}{(\beta + \alpha)(\beta - \alpha)}$$

$$= \frac{\beta + \alpha}{\beta - \alpha} = \frac{1 + \epsilon}{1 - \epsilon}. \quad (2)$$

That is to say, the sectional area of the fluid stratum which would be acted upon will be—

$$\frac{1 + \epsilon}{1 - \epsilon} \kappa A$$

Now, we may evidently regard the aerofoil, with its accompanying peripteral system, as the *equivalent of the hypothetical device which we have temporarily assumed.* The peripteral system actually constitutes a kind of tool or appliance by which the aerofoil is able to deal in effect with more air than actually comes within its sweep. This extended "sphere of influence" of the aerofoil will be termed the *peripteral zone*, and its cross-section, $\frac{1 + \epsilon}{1 - \epsilon} \kappa A$ is the *peripteral area.*

§ 211. The Screw Propeller: Number of Blades.—The number of blades in a propeller must be determined by the conditions of their non-interference. It is evident that if the *peripteral areas* of adjacent blades overlap, the total amount of fluid operated upon will be insufficient and the efficiency must diminish. We must therefore secure that the helices on which the different blades are based are separated *in effect* by an area, measured on a helical surface at right angles, equal to or greater than the peripteral area.

Now this is not an altogether clear proposition, for we are lacking definite information as to the distribution of the peripteral area, and it is evident that we might conceivably have overlapping in one place and clearance at other places. Moreover, the peripteral zone is not in reality a *clearly defined* region such as, as a matter of convenience, we have supposed. On the other hand, we only require an approximate solution; for even if we could gauge to a nicety the spacing of the blades required, we could not take advantage of our knowledge, for we are confined to whole numbers: we cannot employ fractional blades.

We will assume that if a propeller is designed so that no interference is to be anticipated at about the region of greatest efficiency, say 45 degrees, then no interference will take place at all. Furthermore, *we will assume* that the maximum thickness of the peripteral zone can be expressed in terms of the *length of the blade* according to the expression—

$$\frac{\frac{1+\epsilon}{1-\epsilon}\kappa.}{n}$$

Referring to Table XIII., in which values are given as calculated from the *plausible values* of κ and ϵ, for $\frac{1+\epsilon}{1-\epsilon}\kappa$ and

$\frac{\frac{1+\epsilon}{1-\epsilon}\kappa}{n}$ we may note that the latter varies from about unity

for an aspect ratio $n = 3$ to about $\frac{3}{4}$ for $n = 8$. Taking the assumed angle of 45 degrees, and converting these into their *circumferential* equivalents, that is, multiplying by $\sqrt{2}$, we have 1·4 and 1·05. If we presume that the propeller is of the customary proportions, based on a 95 per cent. discard as to diameter, pitch, etc., the length of the blade (in the sense here employed) is approximately twice the radius at the point chosen, so that, expressing the circumferential spacing of the blades in terms of the radius, we shall have for n = 3, radius × 2·8, which, multiplied by $\frac{180}{\pi}$, gives 160 degrees apart, or two blades nearest whole number, and for n = 8 we get 2·1, which gives 120 degrees, or three blades almost exactly.

In view of experience, it is evident that these results are lower than necessary; it is found advantageous to employ more blades than here stated. This discrepancy is perfectly explicable, for the assumptions we have made all tend towards a minimum value. The desirability of keeping the propeller circle as small as possible is probably responsible for the employment of a *fourth* blade in the marine propeller. Four blades doubtless give rise to some slight interference and loss of power, but not sufficient to be seriously detrimental.

If we take three blades as a standard for the marine propeller where n = 3, the corresponding value when n = 8 (a probably impracticable proportion) would be four blades almost exactly. Carrying the matter further on the same basis, if we design an aerial propeller, discarding below 95 per cent. of maximum, the blade length will be approximately 1·2 times the radius (at 45 degrees), so that the proportional number will be 5 blades for n = 3, and 6½ blades for n = 8; that is to say, six blades can be carried. If we lower the discard point to 90 per cent. the conditions as to number of blades will become approximately the same as for the marine propeller discarding from the higher percentage; extending the comparison, it would be very difficult to distinguish the one propeller from the other, both being fashioned in accordance with the present theory for the same n value.

It is probable that, owing to the much lighter pressures required, it will be practicable to employ greater values of n in the aerial propeller than in the marine propeller: an aspect ratio of 6 or 8 does not appear to present any difficulty. This being the case, it is highly probable that the aerial propeller in practice may become almost as economical as, if not more economical than, its marine prototype. The whole question depends upon whether the *gliding angle* for the proportions of blade employed in the two types is in favour of the one or the other.

If the aspect ratio employed is as high as that here suggested, and if it is found advantageous to discard from as high a point as 95 per cent., it may with some confidence be predicted that it will be found advantageous to employ as many as six blades.

Table XIII.

n.	$\kappa \frac{1+\epsilon}{1-\epsilon}$.	$\frac{\kappa \frac{1+\epsilon}{1-\epsilon}}{n}$
3	2·85	·95
4	3·45	·86
5	4·13	·82
6	4·69	·78
7	5·29	·75
8	5·98	·748

§ 212. Blade Length. Conjugate Limits.—In §§ 206 and 211 the limits of the blade length have been assumed as determined by the rejection of those portions of the efficiency curve falling below some stated percentage of the maximum, but it has not been demonstrated that this course results in the highest efficiency over the whole blade.

It might, with some show of plausibility, be argued that since the outer portion of the blade has more work to perform, the region of maximum efficiency should be nearer the point of the blade than given by the method suggested, and that therefore the diameter in terms of the pitch should be less than that deduced. Let us investigate the point in question.

We will suppose that, with a certain stated pitch and therefore an efficiency curve of defined scale, we have (in accordance with the Newtonian principle) to operate on a given cross-sectional area of fluid. Then this area is represented by an annulus whose inner and outer boundaries are of the radii of the blade limits.

Let us, firstly, assume the straight line thrust grading curve of Fig. 129, so that the whole of the fluid within the annular area will be uniformly accelerated; and let us regard the thrust

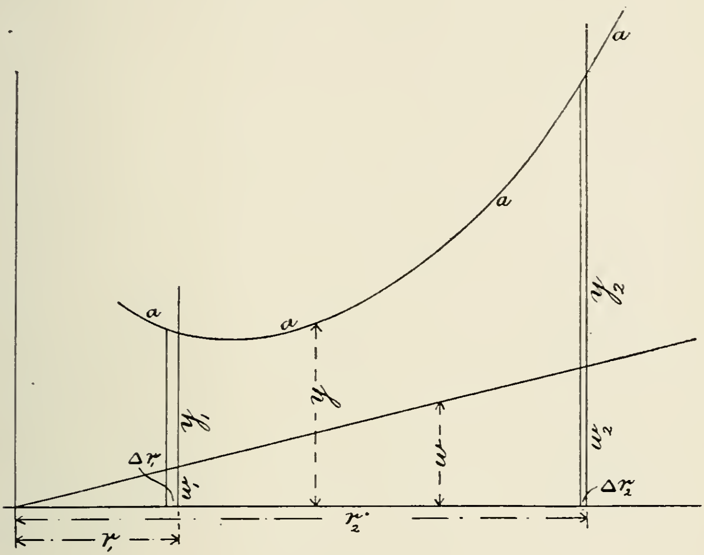


Fig. 132.

grading curve as representing the useful work of propulsion over one unit distance, and let the curve *a a a a* (Fig. 132) represent the *work expended* in the same time. Let E represent the efficiency as a variable, *i.e.*, the ordinate of the efficiency curve, and let w represent the useful work per unit length of the blade, *i.e.*, the thrust grading ordinate (Fig. 132); then the ordinate y of the curve *a a a a* will be $y = \frac{w}{E}$.

Now, if we suppose the limits of the blade length be moved from place to place, so that, however, the annular disc area of the propeller remains contant (which is our fundamental condition), it is a matter of simple geometry to show that the area of the thrust grading curve (of which w is the ordinate) must remain constant; consequently, if we suppose any small variation to take place, and represent the various quantities as given on Fig. 132, we have, for the condition of minimum expenditure of energy—

$$\Delta r_1 \times y_1 = \Delta r_2 \times y_2,$$

or
$$\frac{\Delta r_1}{\Delta r_2} = \frac{y_2}{y_1} = \frac{w_2}{E_2} \div \frac{w_1}{E_1} = \frac{w_2 E_1}{w_1 E_2}.$$

But, since the area of the load grading curve is constant, we have
$$\Delta r_1 w_1 = \Delta r_2 w_2,$$

that is
$$\frac{\Delta r_1}{\Delta r_2} = \frac{w_2}{w_1},$$

so that we have—
$$\frac{w_2 E_1}{w_1 E_2} = \frac{w_2}{w_1}, \text{ or } \frac{E_1}{E_2} = 1,$$

or
$$E_1 = E_2,$$

proving that for the conditions of greatest economy, the blade limits are points of equal efficiency.

Hence, although different proportions may be chosen for the ratio r_1/r_2 (the inner and outer radial limits of the blades), there are *appropriate conjugate values* which are conducive to the maximum efficiency, and these values are determined by the points of equal efficiency on a curve plotted from the equation of efficiency (§ 204, Figs. 126, 127).

§ 213. The Thrust Grading Curve.—We know that the square-ended form of grading curve assumed in the preceding section is inadmissible, and in order that the principle of conjugate blade limits should apply strictly to a real propeller blade we must extend the demonstration to include other forms.

Let us suppose that we regard the thrust grading as made up of a number of small component distributions of load, each of which is strictly in accordance with the hypothesis (Fig. 133). Then we can approximate as closely as we please to a smooth curve, such as we know to be essential, by employing a great number of distributions of individually small magnitude.

But it is evident that each component distribution should in itself obey the conjugate law, hence we may formulate a rule for laying out the load grading curve as follows:—

On the efficiency curve (Figs. 126, 127), cut off by a horizontal line, the portion of the curve defined by the maximum blade limits selected.

Divide the maximum ordinate of the part so cut off (Fig. 133) into some convenient number of equal parts, and draw hori-

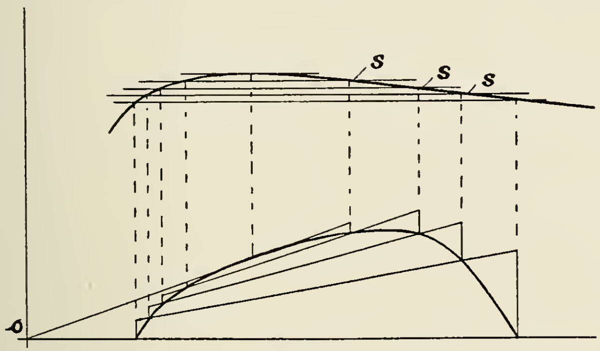


FIG. 133.

zontal lines cutting the efficiency curve at *s s s*. Draw a number of lines through the origin 0, making small angles with the axis of x and with one another, to represent the component increments of the thrust grading distribution, and drop perpendiculars from the points *s s s* to indicate the limits of each increment.

Through the intersections of the perpendiculars let fall, and the inclined lines drawn through the origin, draw the thrust grading curve.

It will be noted that the angular increments of the thrust distribution (the angles between the lines passing through the origin) need not be equal to one another; it may be considered within the province of the designer to vary these as he may think fit. In general, owing to the desirability of utilising as uniformly as possible the whole of the fluid, the angle allotted to successive increments will be diminished as shown in the figure ; the loss of efficiency resulting from modifying the curve in this manner, or even of making serious departures from the theoretical curve, is probably microscopic. The theory as here presented will prove of more value to the designer by letting him realise *exactly what he is doing*, than from its too rigid and literal application.

§ 214. **On the Marine Propeller.**—The marine propeller, which has already been made the subject of comparison, is of especial interest as representing in a concentrated form the experience of over half a century. Beyond the advantages to be gained from an examination of established practice as a quantitative check on our investigation, there is some reasonable probability that we may arrive at facts having a bearing on the future evolution of screw propeller design.

The conclusions to which theory has led have in many cases been already tested by appeal to experience, with, on the whole, satisfactory results, but we have so far made no comparison on the very vital subject of pressure-velocity relationship.

The pressures proper to the conditions of least resistance (minimum gliding angle), given in Tables X. and XII., will be very much greater where water is concerned, owing to the greater density. On the other hand, the coefficient ξ is less, which will have a slight effect in the opposite direction.

The form of blade employed in the marine propeller is of necessity confined to low values of **n** ; long slender forms such as may be well suited to an air propeller will be too weak (unless made of disproportionate thickness) to stand the pressure required. In practice the value of **n** employed is less than 3, usually very much less ; we will therefore confine our attention to blades of this proportion with the knowledge that the results as to pressures and efficiency ought in general to be higher than those that obtain in practice.

Referring to § 181, we have the angle β given by the expression $\sqrt{\dfrac{2\,\xi\,C}{\kappa\,(1-\epsilon^2)}}$. Taking ξ for water as = ·01 and density = 64 lbs. per cubic foot we obtain, β = ·132 (radians) or, = 7·6°, and the theoretical minimum gliding angle is 3·95°, which in practice becomes 6° about. The P_2/V^2 relation (given for air in Table IX.) will become $\rho\,\kappa\,(\epsilon+1)\,\beta = 12{\cdot}5$.

The above are the data on the pterygoid basis; similarly on the *plane* basis, that is, for blades of perfect helical form, we have, taking $\xi = {\cdot}005$, on the principle explained in § 182, $\beta = {\cdot}048$, or, β° 2·75°, that is γ (least value) = ·096, or $\gamma^\circ = 5{\cdot}5^\circ$ The P_3/V^2 relation is given by the expression—

$$\frac{P_3}{V^2} = c\,\beta\,C\,\rho \text{ (§ 186)} = 4{\cdot}55.$$

The above results may be tabulated as follows:—

Pterygoid basis.	β°	7·6°
	γ° (calculated)	3·95°
	γ° (probable)	6°
	$\dfrac{P_2}{V^2}$	12·5
Plane basis.	β°	2·75°
	γ°	5·52°
	$\dfrac{P_3}{V^2}$	4·55

The P/V^2 values given above are in absolute units. The pressure value is extended in Table XIV. as *pounds* per square foot for values of V in feet per second.

§ 215. The Marine Propeller (continued).—Cavitation.—It is very questionable to what extent the *plane* basis of operation is applicable in the case of a screw propeller. There seems to be a grave theoretical objection that does not exist when the motion is rectilinear, as in the analogous case of a weight supported by an aeroplane.

When we have to deal with the propeller blade on the assumption of a discontinuous system of flow, it is evident that the fluid in the dead-water region must be following the blade in its spiral path and must consequently be subject to centrifugal force. So much is this the case that it is hardly possible to conceive of the same fluid remaining in the dead-water region for any considerable length of time, so that the resistance on this basis will be very greatly augmented, and it will not be fairly represented by the figures deduced from rectilinear theory.

This objection does not apply to the calculations made on the

TABLE XIV.

Pressures Proper to Greatest Efficiency for Blade Velocity = V.

V.	Pterygoid Basis.	Plane Basis.
20	156	56
30	351	126
40	624	224
50	975	350
60	1404	504
80	2496	896
100	3900	1400
120	5616	2020
140	7650	2740

pterygoid basis, or at the most only to a very small extent; the author is therefore disposed to think that the pterygoid basis is that indicated by the conditions as giving the correct data for propeller design, the alternative not having the importance that attaches to it in the main problem of flight.

Now the pressure values given in Table XIV. represent not only the pressures appropriate to different blades moving with the mean velocities given, but the pressures appropriate to different portions of the same blade according to its velocity at different points along its length. Moreover, the pressures are not definite *plus* values but represent the pressure difference between the face and back of the blade. But we know that under real conditions a fluid is only competent to bear a certain *maximum negative* pressure (that is, a certain absolute minimum) without giving rise to *physical discontinuity;* that is to say, the formation of a void. Consequently there will be some critical pressure which cannot be exceeded without destroying the peripteral system of flow. The velocity at which this critical pressure is reached marks the limit of speed at which full propeller efficiency can be obtained; for speeds involving any higher velocity the design of a screw propeller becomes a compromise.

The production of a physical discontinuity by the screw propeller was discovered by Messrs. Thornycroft,[1] and is termed "cavitation"; the phenomenon is one that has given considerable trouble to naval architects where high speeds have to be developed. We will endeavour to estimate the critical value of V and show what modifications of design are indicated when the said critical value is exceeded.

We do not know definitely how much of the total reaction is carried as pressure on the face, and how much as vacuum on the back of the blade, but assuming pterygoid form and neglecting the effect of *thickness,* it is probable that the reaction is equally divided. The influence of thickness will be to superpose a *streamline system of flow* on the peripteral system which will result in a general *diminution* of pressure on both faces, so that the reaction will be *more than half* borne by the vacuum on the back of the blade. If the peripteral system comprises any discontinuity it is probable that this will tend in the opposite direction.

On the whole, it is perhaps best to assume the *equal division of the reaction;* and in making this assumption, to bear in mind that if the blade is of heavy section, cavitation will probably commence at a lower velocity than that which theory leads us to expect.

Let us assume that the propeller is working under a head of 2 feet of water in addition to the atmospheric pressure, that is, let us take the total pressure to be 16 pounds per square inch.

[1] Trials of destroyer *Daring.*

Converting this into square foot units, we have permissible vacuum 2,300 pounds per square foot, or maximum total reaction = 4,600, which corresponds to a velocity of approximately 108 feet per second. If we take the pitch as equal to $1\frac{1}{4}$ times the diameter, this velocity, at the extremity of the blade, corresponds to 39·7 feet per second for speed of vessel, that is, approximately, 27 miles per hour or $23\frac{1}{2}$ knots. In practice, the blade of a propeller is never brought to a point as we are now supposing; the end of the blade is always rounded and the pressure consequently less than contemplated by our theory. It is probable that from this cause cavitation does not commence to give trouble till a somewhat higher speed is reached.

It is evident that at speeds above the cavitation limit the design will need modification. It will be necessary to give an area to the blades in excess of that proper to greatest economy, the additional area being required first at the tip of the blade, and as the speed becomes higher the blade will become affected over a greater portion of its length. The secondary consequences of this will be that it will no longer pay to employ the outer blade extremities, and the diameter of the propeller in terms of its pitch will have to be diminished; in ordinary parlance, the screw will become of quicker pitch. Beyond this the aspect ratio of the blades will cease to be of the same importance, since we are unable to employ the higher pressures to which the greater values of $\mathfrak{n}$ give rise. We may therefore expect to find the blade form becoming of more compact outline as higher speeds come into vogue.

It is manifest that for marine work at high speeds it is impossible to construct a fine pitch propeller to give any reasonable economy, for at a comparatively low vessel speed the velocity of the blades will begin to exceed the limiting value, and it will be necessary to add so much extra surface to reduce the pressure that the efficiency will be poor.

In aerial propellers we are fortunately not concerned with the phenomenon of cavitation.

§ 216. **The Influence of the Frictional Wake.**—It has already been shown that the efficiency of a propeller of any kind is increased by the fact of its operating on the frictional wake.

The efficiency that we have been discussing, the E of the screw propeller, represents the efficiency on the basis of § 198; that is to say, the propeller is supposed to act on virgin water, and the *towing* efficiency is that taken as unity.

When we consider the wake as influencing the efficiency we have to adopt a convention. It is known that the wake is in reality a very disturbed region whose velocity varies greatly from point to point. Mr. R. E. Froude has shown that the *mean wake velocity* over the area swept by the propeller may be taken as its effective velocity without serious error, and he has also introduced the useful conception of a *phantom ship* having a speed equal to the actual velocity *minus* the mean wake velocity, that is, $\mathbf{V} - \mathbf{v}$.

The resistance of the phantom ship is supposed, at its velocity $\mathbf{V} - \mathbf{v}$, to be equal to that of the real ship at its velocity $= \mathbf{V}$, so that the propeller designed for the phantom ship on the basis of simple theory will be correct for the real ship to work in its frictional wake. The important fact is thus rendered apparent, that the *form of the propeller proper to highest efficiency is independent of the existence or otherwise of a frictional wake.*

Now the *useful work* is proportional to the actual velocity of the vessel, since the resistance (and therefore the thrust) is the same in both the *phantom* and the *reality;* consequently the useful work is in the relation—

$$\frac{\text{Real}}{\text{Phantom}} = \frac{\mathbf{V}}{\mathbf{V} - \mathbf{v}_1}.$$

But the total work done in propulsion is the same in both cases, therefore if E_1 is the efficiency under real conditions, we have—

$$E_1 = \frac{\mathbf{V}}{\mathbf{V} - \mathbf{v}_1} E.$$

The value of $\mathbf{v}_1$ depends upon the lines of the vessel and position of the propeller, the essential point being the extent to which the frictional wake is led into the periptery of the propeller blades. It is manifest that v_1 is limited to a value less than the sternward component of the impressed velocity (compare § 200).

In the hypothetical case chosen in § 200, where it is assumed that the *whole of the wake current* is utilised by the propeller, we have—

Phantom ship velocity $= V - v_1$, or by § 198—

$$E = \frac{V - v_1}{V - v_1 + \frac{v}{2}} = \frac{V - v_1}{V - \frac{v_1}{2}}, \quad \text{(for } v_1 = v\text{,)}$$

but for *real* ship $$E_1 = \frac{V}{V - v_1} E,$$

or $$E_1 = \frac{V}{V - v_1} \times \frac{V - v_1}{V - \frac{v_1}{2}} = \frac{V}{V - \frac{v_1}{2}},$$

which is the result already deduced in § 200 by the direct application of the Newtonian principle.

The device of the *phantom ship* is in reality merely a method of expressing a simple problem in relative motion in a palatable form; it is obvious that the argument treats the wake current as a favourable tidal current, or as the flow of a river, the ship's motion being credited in respect of its change of position relatively to some fixed mark; the method of the "phantom ship" presents the problem in a clear and precise form.

The question of wake influence is probably of less importance in connection with aerial flight than it is in the problem of marine propulsion.

§ 217. The Hydrodynamic Standpoint. Superposed Cyclic Systems.—It is of interest to form a mental picture of the hydrodynamic system of flow that accompanies a screw propeller.

It is evident that according to peripteral theory each blade of the propeller forms the axial core of a cyclic system and that the necessary condition of multiple connectivity is carried out by vortex filaments containing rotation trailing from the ends of each blade.

There will need to be as many cyclic systems as there are blades, so that if there are n blades the region will require to be n-ply connected. In the case, for example, of a two-bladed propeller the two cyclic paths are represented in Fig. 134 by

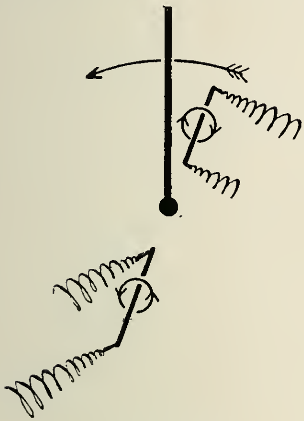


Fig. 134.

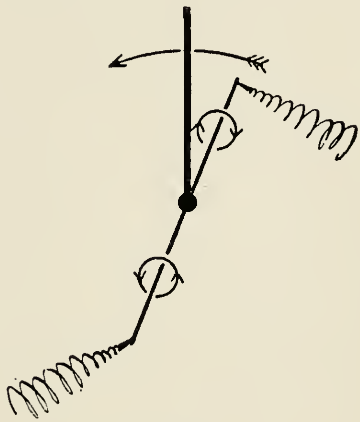


Fig. 135.

the two circuits drawn round the blades; the trailing vortices are shown diagrammatically.

It is possible that the inner end vortices are unnecessary, for the boss and shaft may be found to determine the connectivity of the region at the axis end of the blades (Fig. 135).

The vortex filaments are presumed to persist in the region of the wake till they have, metaphorically speaking, "taken root" in the fluid, so that the conditions of multiple connectivity are simulated. It would appear to be only necessary to suppose *rotation* to become generally distributed (through the agency of viscous stress) in the wake of the propeller to bring about the necessary condition. (Compare Chaps. III. and IV.)

In propellers giving rise to cavitation, or when air is sucked down owing to insufficient immersion, the dependent vortices become visible by their empty cores, and may be seen as interlacing helices following in the track of the extremity of each blade, like adherent strings of sea-weed.[1]

The vortices from the external extremities of the blades are all of the same "hand" and consequently tend to wind round one another; they may be conceived to break up into spiral groups and perhaps sub-groups, as they are left behind in the propeller race, after the manner indicated already in the case of the aerofoil (Fig. 86).

§ 218. On the Design of an Aerial Propeller.—A few simple rules may be formulated for the design of an aerial propeller; these rules will be applicable *mutatis mutandis* to the marine propeller.

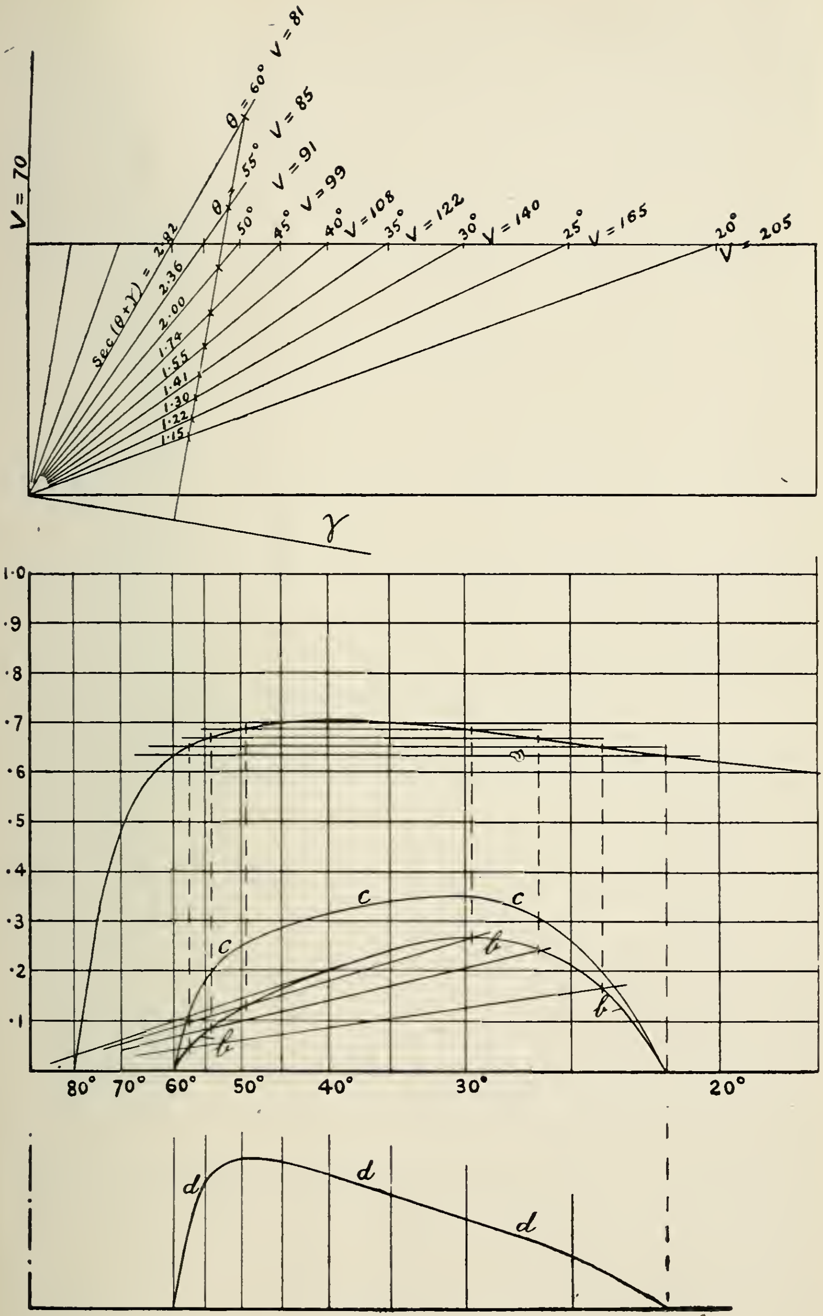
(1) From the conditions, assess the probable value of γ (usually about 10 degrees), and (Fig. 136) plot the efficiency curve from the equation (§ 204). Any arbitrary scale may be employed.

(2) Decide on "discard point"; that is, the minimum percentage of maximum available efficiency, and so determine blade length.

(3) Draw the thrust grading curve, *b b b* (Fig. 136), as in § 213 (Fig. 133). At this point the designer has to exercise his judgment; it is perhaps best to draw a trial curve freehand, the object being a smooth curve beginning and ending at zero, but in general character to simulate the truncated wedge form based on the Newtonian theory; then let fall perpendiculars from the conjugate points of equal efficiency, and draw radial lines through the origin to suit the freehand curve as nearly as possible; then correct the freehand curve to pass through the intersections.

(4) From the thrust grading curve *b b b* (Fig. 136) derive the load grading curve *c c c*; the ordinates being calculated by multiplying the thrust ordinates by the corresponding values of sec $(\theta + \gamma)$ (Fig. 136).

[1] The author has, for example, observed such air-core vortices from the after-deck of twin-screw *S.S. New York*.



V = 70
θ = 60° V = 81
θ = 55° V = 85
50° V = 91
45° V = 99
40° V = 108
35° V = 122
30° V = 140
25° V = 165
20° V = 205
Sec (θ+γ) = 2·82
2·36
2·00
1·74
1·55
1·41
1·30
1·22
1·15
γ
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
80°
70°
60°
50°
40°
30°
20°

FIG. 136.

(5) Calculate the values of V^2 for different points along the blade (§ 207), and *divide* the values of the ordinates of the load grading curve *c c c* by the corresponding values of V^2, and draw a curve representing the *quotient*. This is the *linear grading curve d d d* and represents the relative height of the arched section at every point along the blade. (Compare § 192.)

(6) The "plan form" or "development" of the blade may now be laid out. If we proceed on the lines indicated by present theory, the plan form will be everywhere proportional to the linear grading; thus we have to settle the *aspect ratio* of the blade, lay off the maximum width, and draw a curve whose ordinates from point to point are proportional to the linear

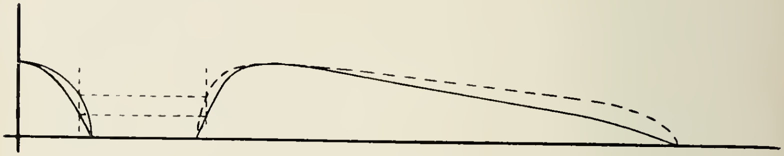
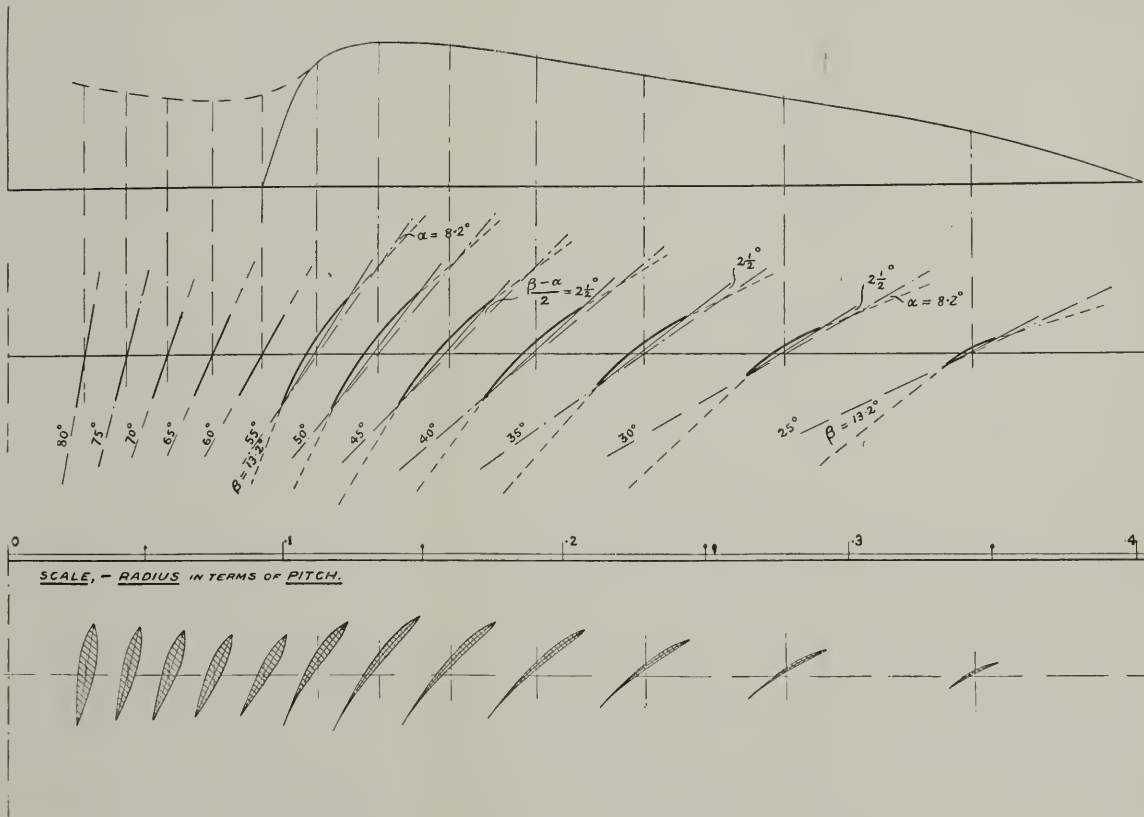


Fig. 138.

grading ordinates (Fig. 137 (*a*)). If we adopt this design of blade the sectional *form* will be constant throughout the length, varying only in its *scale;* that is to say, the materialised α and β angles will be everywhere the same (Fig. 137).

The theory may possibly be incomplete; as discussed in §§ 190, 191, 192, etc., there may be some unformulated objection to the pointed extremities to which present theory gives rise. If this is the case the section will become flatter towards the extremities, the linear grading remaining the same and the width of the blade becoming greater. If we take the elliptical aerofoil as our model we may derive the corresponding blade form by the construction given in Fig. 138, elliptical ordinates being substituted at every point for the corresponding parabolic or segmental ordinate.

(7) If such a modified plan form is adopted the sectional form



α = 8·2°
(β − α)/2 = 2½°
2½°
2½°
α = 8·2°
80°
75°
70°
65°
60°
55°
β = 13·2°
50°
45°
40°
35°
30°
25°
β = 13·2°
0
·1
·2
·3
·4
SCALE, — RADIUS IN TERMS OF PITCH.

Fig. 137.

of the blade should be designed at every point to suit the width and grading, the angles α and β having appropriate values assigned from point to point, according to the value of the constant ϵ. It is possible that the constant ϵ ought to vary from point to point along the blade; if this is so it is a matter on which we have so far no information; for the present it should be taken as the aerofoil value of ϵ proper to the value of n employed.

In Fig. 137 the blade is supposed continued to connect to the boss. Such continuation is always necessary, unless a boss of very large diameter is employed, the continuation being of stream-line section symmetrically disposed about the pitch helix. It will be observed that the linear grading falls with extreme rapidity as the inner "extremity" of the blade is approached, and thus the change of form from the pterygoid to the symmetrical stream-line section is very abrupt. It is probably advantageous to carry the pterygoid section beyond the theoretical blade limit and so merge it more gradually into the simple form. No cognisance of this structural feature has been taken in the hypothesis.

It should be remembered that the $\theta + \gamma$ spiral is at every point the analogue of the horizontal line, and the one from which the α β angles are laid off; on this basis the setting out of the section is the same as for an ordinary aerofoil, but the full "dip" forward can be given to the section, since the possibility of the loss of equilibrium is no longer a factor (§ 138).

Parenthetically it may be remarked that the $\theta + \gamma$ series of spirals does not form a helix, for the angle γ is constant at all points. This would usually be expressed by saying that the pitch of the blade *increases* towards the tip, but we know that the ordinary manner of defining the pitch by the mean angle of the blade is unscientific.

(7) Number of blades. The determination of the maximum number of blades permissible has been discussed in § 211, and it has been shown that this depends but little upon the

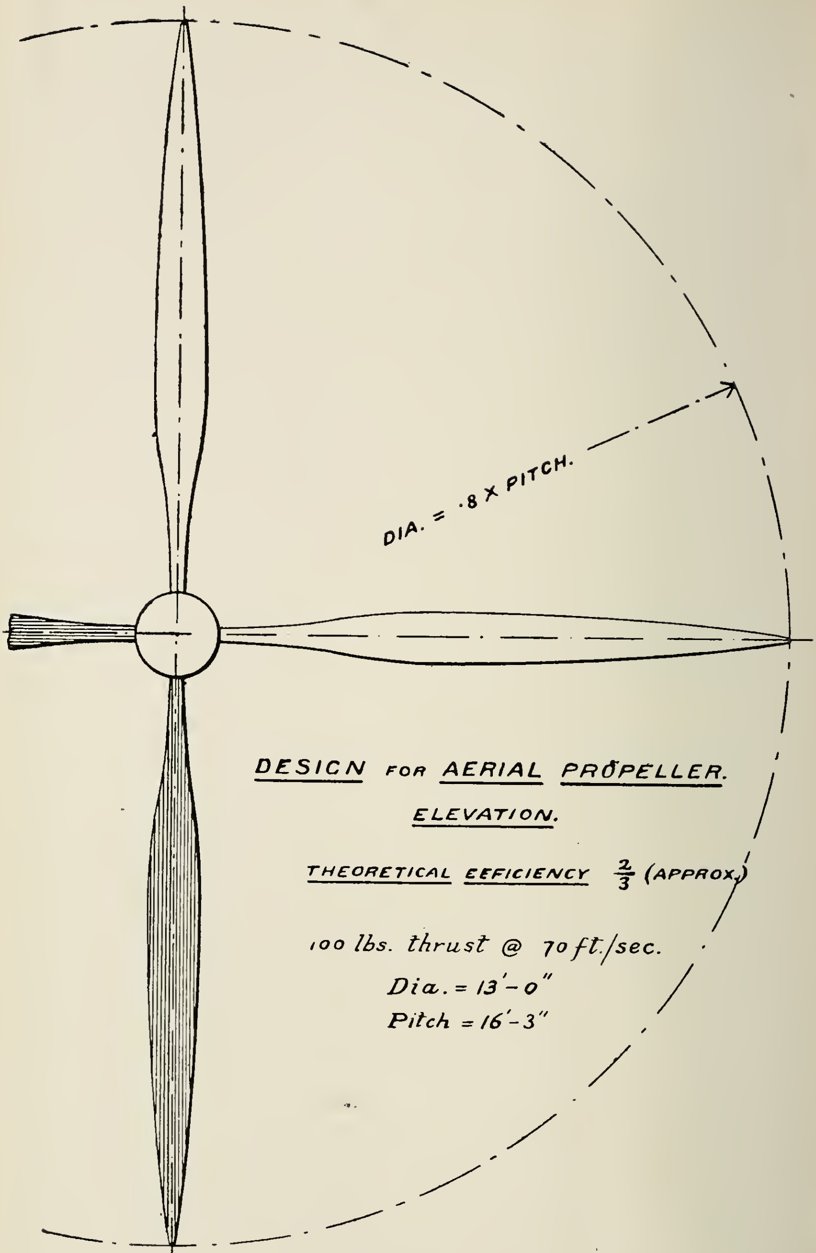

DIA. = ·8 X PITCH.
DESIGN FOR AERIAL PROPELLER.
ELEVATION.
THEORETICAL EFFICIENCY 2/3 (APPROX.)
100 lbs. thrust @ 70 ft./sec.
Dia. = 13′-0″
Pitch = 16′-3″

Fig. 139.

value of "n," being chiefly dependent upon the relative length of the blade as compared to the diameter; the number of blades thus depends upon the *discard* percentage. For a 90 per cent. discard four blades are the appropriate number; if the discard be 95 per cent. six blades may be employed. In general the following rough-and-ready rule may be employed: Let r_1 and r_2 be the radii of the inner and outer extremities of the blades, then the number of blades permissible will be $= 2{\cdot}5\,\frac{r_2 + r_1}{r_2 - r_1}$, fractions being neglected.

(8) All that remains to be done is now to give a *scale* to the design that will render the propeller suitable for the intended load. To this end any convenient scale should first be assumed and the load calculated, for the actual value of V (the velocity of flight), which it is intended to employ. The linear dimension will then be in the ratio to the dimension required as the square root of the calculated thrust is to the square root of the thrust required. That is to say, the *scale unit* will be in the inverse ratio.

An example of the design of an aerial propeller on the foregoing principles is given in Figs. 136, 137, 138, 139; the supposed data being as follows:—Velocity 70 feet per second; thrust = 100 lbs.; discard 90 per cent.; n = 6; γ taken = 10 degrees.

§ 219. Power Expended in Flight.—The principles governing the expenditure of power in flight have, in this and the preceding chapters, been fully expounded, and it now only remains to draw certain elementary deductions.

The power essential to flight or *thrust horse-power* may be defined as that represented by the thrust multiplied by the velocity of flight, that is to say, the equivalent of the *tow-rope expenditure.* The actual power required will then be a *thrust or essential power* divided by the *efficiency* of the *propulsion.*

We have seen (§ 200) that the efficiency of propulsion may theoretically be *greater than unity,* so that the term *essential* must not be construed as meaning the *minimum* theoretically necessary on the Newtonian basis.

Now, the essential power will be given by the expression $W \gamma V$, where γ (assumed to come within the definition of a small angle) is expressed in circular measure. Now, if γ is constant in respect of V, as has been proved to be the case so long as the *body resistance* is regarded as negligible, or separately computed, the following deductions may immediately be made :—

(1) The *energy* required to *travel from point to point* is independent of the velocity and *is constant.*[1]

(2) *The power* (horse-power) required is *directly as the velocity.*

(3) From (1) it follows that the maximum *range of flight* of a flying machine must depend upon the fuel carrying capacity, the energy value of the fuel, and the total efficiency of the prime mover and propelling mechanism, and is independent of the speed of flight.

(4) From (2) it follows that the velocity of flight is limited by the relation of horse-power to weight, and, other things being equal, is proportional to the horse-power per unit weight of the prime mover.

These conclusions are of considerable importance, and are illustrated in the Tables as follows :—

Table XV., column (1), gives, for values of $\gamma^\circ = 6^\circ, 7^\circ, 8^\circ, 9^\circ$, and 10°, the distance that could be run if an aerodrome had at its disposal the total energy of its own weight of *hydrogen* taken as giving 48,000,000 foot lbs. per lb. Column (2) gives the same information for petroleum spirit, taken as equal 16,000,000 foot lbs. per lb.; column (3) is based on the assumption that 25 per cent. only of the total heat is available, as representing the thermal efficiency of the petrol engine. Column (4) the distance after an allowance of 75 per cent. mechanical efficiency of engine and transmission, and a 66·6 per cent. efficiency of propulsion. Lastly, column (5) gives the actual range, or maximum possible distance, on the basis of columns (2) to (4) on the assumption

[1] It is understood that the proportions of the aerodrome are varied with changes of velocity to comply always with the conditions of least resistance, §§ 165, 176, 185.

that the aerodrome or flying machine carries 10 per cent. of its *mean* weight as fuel, *i.e.*, petroleum spirit. The mean weight is, under these conditions, 5 per cent. less than the initial and 5 per cent. more than the final weight, the loss of weight being due to the fuel consumption.

As a maximum estimate of the range of flight conceivably possible without some fundamental discovery in fuel and prime movers we may take the following supposititious case. Using liquid hydrogen as fuel, and carrying 25 per cent. of the total mean weight, and assuming a yet-unheard-of thermal efficiency of 50 per cent., a total mechanical efficiency of 90 per cent., and a propeller efficiency of 70 per cent., with the minimum angle of γ given in the Table (= 6°), the exhaustion of fuel will be complete after a flight of 6,800 miles distance.

The above estimate is based on an assumed development of the heat engine and other mechanical refinements not yet within sight, and indeed such as may never be realised. If we confine ourselves to existing appliances and existing methods it is doubtful whether the maximum range of flight can (without devoting the whole resources of the machine to the carrying of fuel), ever exceed 1,000 miles, and for the present this may be regarded as the probable extreme outside limit.

Table XV.

Possible Range of Flight on Basis of Computation given in Text.

(Column 5 gives computed range in miles, assuming propulsion by petrol motor and screw propeller, for fuel capacity equal one-tenth of total weight.)

γ_0.	γ.	(1.)	(2.)	(3.)	(4.)	(5.)
6°	·105	86,600	28,866	7,216	3,608	360
7°	·122	74,500	24,833	6,208	3,104	310
8°	·140	65,000	21,666	5,416	2,708	270
9°	·157	57,900	19,300	4,825	2,412	251
10°	·175	52,000	17,333	4,333	2,166	216
11°	·192	47,400	15,800	3,950	1,975	197
12°	·210	43,300	14,433	3,608	1,804	180

§ 220. **Power Expended in Flight (continued).**—It has been shown that, neglecting body resistance, the power per unit weight requires to increase directly as the velocity. Table XVI. gives the calculated indicated horse-power *per* 100 *lbs. weight* for velocities ranging from 15 to 100 feet per second, the Table also being figured for the thrust horse-power for velocities from 30 to 200 feet per second. The value taken for the total efficiency is the same as employed in calculating column (4) of the preceding Table, *i.e.*, 75 × 66·6 per cent. = 50 per cent., so that the velocities for a given horse-power value are in the ratio of 2 : 1.

The calculation has been made for values of γ extending from 6 degrees to 12 degrees, as in the preceding Table; experiment would appear to show that 10 degrees is as low a value of γ as can be obtained under practical conditions; it is, however, possible that with increased experience lower values may be obtained. An aerodrome whose γ is greater than 12 degrees is certainly of inefficient design.

Table XVI.

Indicated Horse-power and Thrust Horse-power per 100 *lbs. weight at different Velocities and for different Values of* γ.

	Indicated horse-power per 100 pounds weight at velocities (ft./sec.)—												
γ_o.	15.	20.	25.	30.	35.	40.	45.	50.	60.	70.	80.	90.	100.
6°	0·57	0·76	0·95	1·14	1·33	1·53	1·72	1·91	2·28	2·66	3·06	3·4	3·8
7°	0·66	0·87	1·11	1·33	1·55	1·77	2·00	2·22	2·66	3·10	3·54	4·0	4·4
8°	0·76	1·02	1·27	1·53	1·78	2·04	2·29	2·51	3·06	3·56	4·08	4·6	5·1
9°	0·85	1·14	1·42	1·71	2·00	2·28	2·56	2·86	3·42	4·00	4·56	5·1	5·7
10°	0·95	1·27	1·59	1·91	2·22	2·54	2·86	3·18	3·82	4·44	5·08	5·7	6·3
11°	1·05	1·40	1·75	2·10	2·44	2·80	3·14	3·50	4·20	4·88	5·60	6·3	7·0
12°	1·14	1·53	1·91	2·29	2·67	3·05	3·44	3·82	4·58	5·34	6·10	6·9	7·6
	30	40	50	60	70	80	90	100	120	140	160	180	200
	Velocities (ft./sec.) at which thrust horse-power is required as above.												

If we take account of *body resistance*, we know that the total resistance is greater the higher the velocity, for the body resistance increases as V^2, so that the angle γ is no longer constant in respect of V. We also know (§ 171) that the influence of the weight of the aerofoil, as additional to the load carried, is to place a lower limit on the velocity that may be usefully employed.

The Equation (5) of § 171 gives the condition of least resistance. The value of γ can thus be calculated for any set of conditions, and the power data obtained from the Table.[1] By plotting from the equations the conditions other than for least resistance may be examined with equal facility and the γ values determined.

Before leaving the subject of power expenditure it is desirable to point out the extent to which the future of flight and the uses of a flying machine are circumscribed by economic considerations.

Leaving all attendant difficulties on one side, it is evident that the conveyance of goods by flying machine would be comparable, so far as power expenditure is concerned, with drawing them on a sleigh over a common road, so that where any other method of transport is possible, flight may be regarded as out of the question. In addition to this, the range of a flying machine must, unless after the manner of a soaring bird it derives its energy from wind pulsation, be strictly limited to a few hundred miles between each replenishment of fuel; and consequently we cannot at present regard aerial flight as a means of ocean transport, or even as a means of exploring inaccessible regions where the distance to be accomplished exceeds that stated.

Beyond this the velocity of flight is limited by the horse-power weight factor. If, as an example, we suppose that 25 per cent. of the weight of the machine is taken up by the motor itself, and if the motor weigh only $2\frac{1}{2}$ lbs. per horse-power, it is improbable, taking everything into account, that seventy miles per hour can be

[1] In §§ 181 and 189 it has been pointed out that the γ value, independently of body resistance, is in practice greater than can at present be deduced from pure theory.

exceeded. Flying is comparable to hill climbing on a road automobile where γ represents the gradient, this being about 1 in 5 or 1 in 6, and where the transmission efficiency is limited to about 66 per cent. The velocity limit rests entirely on the weight per horse-power, the aerodrome being presumed designed for least resistance ; any continued improvement in *prime movers*, tending to a reduction of weight, will react in the direction of rendering higher aerial speeds practicable.

If it should be found possible to "soar" on a large scale after the manner of an albatros or gull, the limitation of range may, in certain exceptional cases, be partially or wholly removed.

It may be noted that on the liquid hydrogen estimate of maximum possible range, no allowance has been made for the possible power to be derived directly from the expansion prior to combustion. We have also omitted to discuss the possible increase of thermal efficiency theoretically available by the employment of the low temperature of the boiling point of hydrogen as a *refrigerator*, that is, as the temperature at which the heat engine discards. The use of liquid hydrogen is at present too daring and distant a suggestion to be taken quite seriously ; it has here been put forward merely as representing the maximum known fuel value in a possibly available form.

CHAPTER X.

EXPERIMENTAL AERODYNAMICS.

§ 221. **Introductory.**—Experimental aerodynamics must at present be regarded as in its infancy. The methods employed up to date have not yielded results of an exactitude comparable to that readily obtainable in other branches of physical science.

There are in the main three methods of investigation open. Firstly, experiments upon planes or other bodies propelled through still air, the subject of experiment and the measuring appliances being mounted either on the arm of a *whirling table* or on the front of a locomotive vehicle. Secondly, the measurement of the reactions produced by a fluid in uniform motion on a fixed body. Thirdly, by measurement and deduction from experiments in free flight.

The first method is that most generally adopted, admirable work having been done in this direction by Dines, Langley and others. The second method has been used to some extent by Dines, and the third (the method of free flight) has been developed to a certain extent by the author.

The earlier experimenters, Robins (1761), Hutton (1787), and Vince (1794—5, 1797—8), employed a primitive form of whirling table, the invention of which is attributed by Hutton to Robins, and no earlier record appears to exist of the employment of this device for the purpose contemplated. The whirling table as known to Hutton is represented diagrammatically in Fig. 140, in which a horizontal arm A is mounted on a vertical axis B, which is caused to rotate by the silk cord C and weight D; the body on which experiments are to be made is mounted at the extremity of the arm as at E, and the determination of its resistance is made by altering the weight D until a certain definite speed of rotation is attained. It is evident that, knowing the length of the arm, the diameter of the axle on which the cord is wound, and the weight employed, the calculation of the resistance is a matter of simple arithmetic. The resistance of the body under

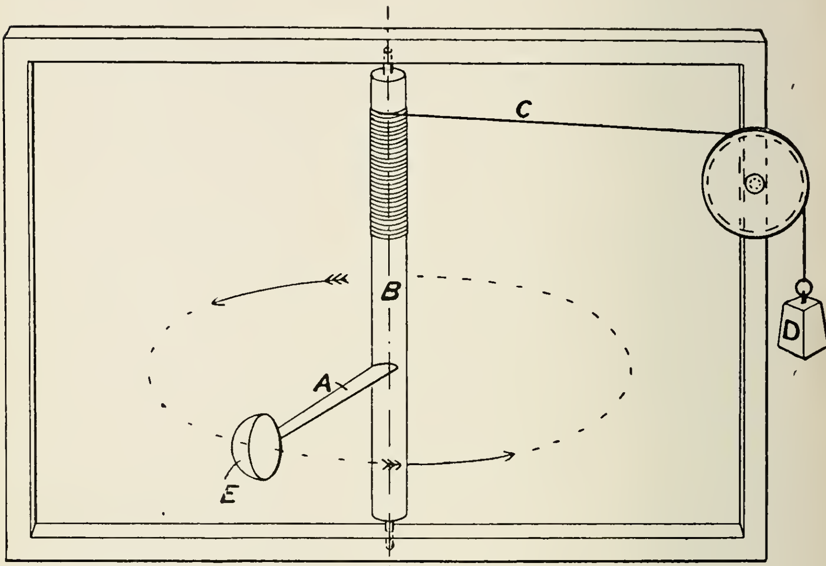


Fig. 140.

investigation only forms part of the total resistance, and a preliminary experiment is necessary to determine the resistance proper to the apparatus itself. Precautions were usually taken to prevent so far as possible frictional resistance, the variations of which would otherwise give rise to error of sensible magnitude.

§ 222. Early Investigations. Hutton—Vince.—One of the objects on which considerable experimental attention was focussed at an early period was the investigation of the solid hemisphere in plane and in spherical presentation. This is probably due to the fact that this problem forms the basis of prop. xxxiv., Book II., of the "Principia" *as touching the Newtonian Medium.* It was evidently the aim of the early workers to ascertain the extent to which Newton's results would prove applicable to a real fluid.

In the case in question it was found by Hutton that—

(1) The pressure on the hemisphere in either presentation varies, according to the Newtonian law, that is as the square of the velocity; and,

(2) The resistance in *plane presentation* is approximately two-and-a-half times as great as in *spherical presentation*, instead of only twice as great as demonstrated by Newton. This result was subsequently confirmed by Vince, whose relative figures were 1 to 2·46.

These results were considered at the time as a substantial confirmation of prop. xxxiv., or, rather, as showing that the behaviour of a real fluid does not greatly differ from that of the discontinuous medium; probably we have here the reason why many subsequent writers have been misled into assuming the applicability of the Newtonian sine^2 law in the case of the inclined aeroplane. In point of fact the coincidence, had it been far more complete, would be of no significance whatever; the system of flow in actuality bears no resemblance to the dynamic system of Newton.

The fallacy of the sine^2 law was first clearly demonstrated by Vince in his paper (to which reference has already been made); he gives experimental data showing that the resistance *in the line of flight* varies, for small angles, as $\text{sine}^{1\cdot73}$ of the angle; this, if we neglect the influence of skin-friction, corresponds to a pressure normal to the plane varying as the $\text{sine}^{\cdot73}$. The fact that the index here is less than unity can be reasonably accounted for on the assumption that part of the resistance is due to skin-friction, which is roughly constant in respect of the inclination.

§ 223. Dines' Experiments.[1] Method.—Coming now to the modern period, we have to examine two independent series of experiments, made almost simultaneously, respectively by Mr. W. H. Dines in England and by the late Prof. S. P. Langley in America.

To Dines we owe a particularly beautiful and original method of employing the *whirling table* for the determination of aerodynamic data. In all the modern applications of the whirling table, the determination of the resistance of the object of

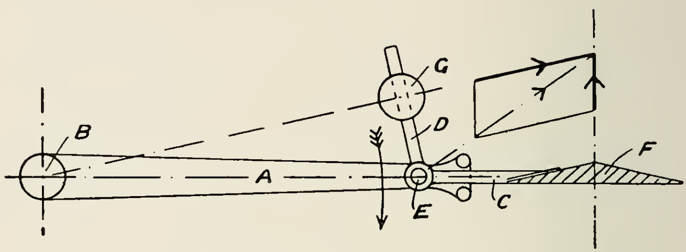


FIG. 141.

experiment is made quite independently of the propulsion of the table, *some form of balance* being employed mounted at the extremity of the rotating arm, the motion of the latter being maintained by the application of a power installation, and the speed of rotation accurately recorded by a chronograph. Dines conceived the possibility of balancing the *aerodynamic reaction*, which varies approximately as the square of the velocity,

[1] In the present account of the investigations of Mr. Dines, the following publications have been consulted :—"Some Experiments made to Investigate the Connection between the Pressure and the Velocity of the Wind" (Dines, *Quart. Journ. Royal Met. Soc.*, Vol. XV., October, 1889), and "On Wind Pressure upon an Inclined Surface" (Dines, *Proc. Royal Soc.*, Vol. XLVIII., 1890). Further particulars of Mr. Dines' experiments of aerodynamic interest will be found in the following:—"Mutual Influence of two Pressure Plates upon each other," and "On the Variations of Pressure caused by the Wind Blowing across the Mouth of the Tube" (Dines, *Quart. Journ. Royal Met. Soc.*, Vol. XVI., October, 1890).

against the *centrifugal force* of an appropriately arranged weight, which varies in like ratio. By this means the measurement of the reaction is made virtually independent of the velocity of flight, so that a possible source of error is avoided and the conduct of experiments is greatly simplified.

The Dines apparatus is illustrated in the form of a rudimentary diagram in Fig. 141, in which A is an arm of the whirling table pivoted at and revolving about the point B, a bell crank lever $C\ D$, delicately centred at E, carries on its two arms respectively the pressure plane, F (or other body whose resistance it is desired to ascertain), and the bob weight G whose centrifugal force forms the measure of the pressure reaction. The condition of equilibrium is that the resultant of the two forces passes through the pivot centre E. It is evident that these two forces, in equilibrium at any one speed, will be in equilibrium for all speeds, for their relative direction undergoes no change, and they are each proportional to the velocity squared and so are proportional to one another. The bob weight D is made adjustable on the arm D and the condition of equilibrium is ascertained by trial. In one modification the instrument is made to perform automatically its own adjustment.

§ 224.—Dines' Method (Mathematical Expression).

Let, A = area of plane.
,, V = velocity of the bob weight.
,, $c_1 V$ = velocity of the centre of pressure of the plane.
,, r = radius of the path of the weight.
,, R = resistance (poundals) acting on the plane.
,, F = centrifugal force of the bob weight (poundals).
,, c_2 = lever ratio, so that $R = c_2 F$.
,, M = mass of bob weight.
,, P = pressure on plane (poundals per square foot.)
,, C = constant in expression, $P = C \rho V^2$ (§ 134).

Then — $P = C \rho (c_1 V)^2$ and $R = A P$

$$\therefore \quad R = A C \rho c_1^2 V^2$$

But— $$F = \frac{M V^2}{r}, \quad \text{and} \quad R = c_2 F,$$

$$\therefore \quad A C \rho c_1^2 V^2 = \frac{c_2 M V^2}{r} \quad \text{or} \quad C = \frac{c_2 M}{A \rho c_1^2 r}$$ [1]

all of which are known quantities ; consequently the value of the constant C is determined.

§ 225. **Dines' Method (continued).**—Of the practical working of the foregoing apparatus, which may not inaptly be termed a *centrifugal balance*, Mr. Dines says :—"It had been assumed in the above that the wind pressure varies as the square of the velocity. The experiments have proved this to be the case, for when upon a calm day equilibrium for any plate is once attained, it has been found impossible to disturb it by any alteration of the velocity of rotation, and since the centrifugal force varies as the square of the velocity the wind pressure must do so also. For the smaller planes the maximum velocity of which the machine is capable is about seventy miles per hour."

We may consequently infer that the V^2 law of resistance holds good as a very close approximation over a very considerable range of speed, certainly as great a range as concerns the problem of flight. This is a result previously considered in doubt.

The arrangement of the centrifugal balance figured is not one altogether suited to planes of large size, owing to the fact that the different portions of the plane are situated at different distances from the centre of the whirler, and the position of the centre of pressure becomes uncertain. In such cases a modified design is adopted (Fig. 142), of which the description is given in § 227.

The obvious difficulty of observing the position of the plate

[1] This expression differs somewhat in form from that given by Dines, the difference being due firstly to the introduction of the *area* (A) of the plane, and the restriction of the use of the term *pressure* as being of the dimensions *force divided area*, and secondly to the difference in the units employed, the use of absolute units eliminating the gravitation constant.

when in rapid motion is overcome electrically, the range of motion permitted to the lever being limited by stops to about one degree, the stops forming at the same time electrical connections to alternative circuits. The two circuits are arranged in connection with a galvanometer so that contact with one stop causes a current in one direction and contact with the other

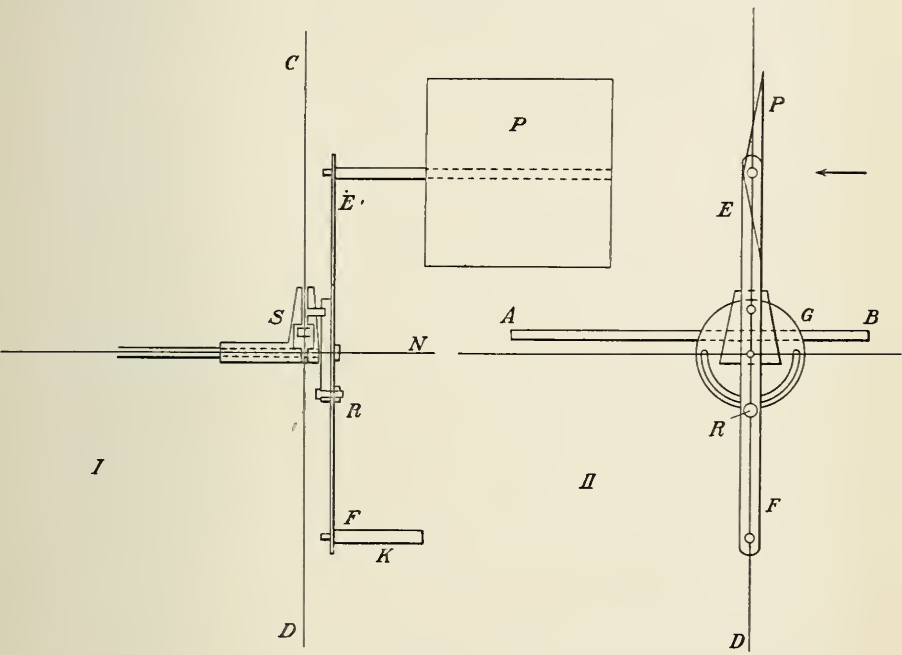


FIG. 142.

causes a current in the opposite direction. When equilibrium is established the galvanometer needle either remains unaffected, or oscillates from one side to the other indicating that the current flows about equally in either direction.

§ 226. Dines' Results. Direct Resistance. — A considerable number of experiments on the resistances of planes and solids of various forms are found summarised in the following table, which is taken intact from the "Experiments on the Pressure and

Velocity of Wind." Many of these results have been already referred to in a previous chapter.

TABLE XVII.

Showing the pressure upon various plates at a velocity of 20·86 *miles per hour. The values are reduced to the standard temperature and pressure. The flat plates were cut out of hard wood* $\frac{3}{8}$ *inch thick. Allowance has been made for the arm which carried them.*

Plates.	Actual Pressure in pounds.	Pressure in pounds per square foot.	No. of Experiments.
A square, each side 4 in.	0·17	1·51	4
A circle, 4·51 in. diameter, same area	0·17	1·51	9
A rectangle, 16 in. by 1 in.	0·19	1·70	7
A circle, 6 in. diameter	0·29	1·47	7
A square, each side 8 in.	0·66	1·48	8
A circle, 9·03 in. diameter, same area	0·67	1·50	12
A rectangle, 16 in. by 4 in.	0·70	1·58	4
A square, each side 12 in.	1·57	1·57	7
A circle, 13·54 in. diameter, same area	1·55	1·55	14
A rectangle, 24 in. by 6 in.	1·56	1·59	6
A square, each side 16 in.	2·70	1·52	6
A plate, 6 in. diameter, $4\frac{3}{4}$ in. thick	0·28	1·45	5
A cylinder, 6 in. diameter, and $4\frac{3}{4}$ in. long	0·18	0·92	4
A sphere, 6 in. diameter	0·13	0·67	8
A plate, 6 in. diameter, with 90 degrees cone at back	0·29	1·49	4
The same, with cone in front	0·19	0·98	4
A plate, 6 in. diameter, with sharp cone 30 degrees angle at back	0·30	1·54	4
The same with cone in front	0·12	0·60	4
A 5 in. Robinson cup, mounted on $8\frac{1}{2}$ in. of $\frac{1}{2}$ in. rod	0·28	1·68	8
The same with its back to the wind	0·12	0·73	4
A 9 in. cup, mounted on $6\frac{1}{2}$ in. of $\frac{5}{8}$ in. rod	0·82	1·75	3
The same with its back to the wind	0·28	0·60	3
A $2\frac{1}{2}$ in. cup, mounted on $9\frac{3}{4}$ in. of $\frac{1}{4}$ in rod	0·13	2·60	3
The same with its back to the wind	0·05	1·04	3
One foot of $\frac{5}{8}$ in. circular rod	0·09	1·71	9

In addition to the experiments tabulated, other resistance experiments are recorded, notably those on perforated plates dealt with in § 143, also the effect of a projecting lip on the edge of the normal plane, § 141.

§ 227. Dines' Experiments (continued). Aeroplane Investigations (Apparatus).—The description and results of these experiments are given in Mr. Dines paper on "Wind Pressure upon an Inclined Surface," (*Proc. Royal Soc.*, Vol. XLVIII.). The apparatus employed is a form of the *centrifugal balance* described in § 223, and the "planes" employed are of the triangular section already illustrated in Fig. 100 (*a*).

Two diagrammatic views of the centrifugal balance employed are given in Fig. 142, which is reproduced from Mr. Dines' paper. In this, like letters refer to like parts in the two views; and we have the pressure plane *P* mounted on the arm *E F* pivoted to turn about the axis *M N*, which is arranged radial to the axis of the whirling table. We have the bar *A B*, which is the mass whose centrifugal force is employed to measure the reaction, mounted slidably in a pivot piece free to turn about the vertical axis *C D*, and the pivot piece and the pressure plane arm are geared together by a stud projecting from the former and engaging with the latter. This may be looked upon as equivalent to a single tooth of an imaginary pair of bevel gears. A counterpoise weight *K* performs the double function of statically balancing the pressure plane, and of rendering the arm *E F* symmetrical in respect of wind pressure.

The adjustment of the bar *A B* by an ingenious device was arranged to take place automatically. This automatic arrangement consists of a windmill carried on the arm of a whirling table, arranged to drive *whenever possible* a crown wheel, by means of a long pinion *engaging with the crown wheel on both sides*. The crown wheel is carried on the pivot piece, and by means of a rack and pinion moves the bar *A B* in the one direction or the other. When the pivot piece reaches either of its extreme positions (the total range being a few degrees), the crown wheel becomes disengaged from its pinion on the one side or the other, the windmill immediately begins to operate, and the bar undergoes displacement in the direction required to restore equilibrium. So long as the balance is perfect and undisturbed, the pivot piece remains in an intermediate position, and the crown wheel is locked by the double engagement of its pinion.

In this centrifugal balance, as in all apparatus of its type, the precise speed at which the whirling table is propelled is unimportant for the reasons already given. It is, however, necessary that the velocity should be steady, *i.e.*, the whirling table should not be undergoing acceleration when the experiment is made. The reason of this precaution is that the pivot piece and bar possess moment of inertia, and *change of speed of rotation involves a torque* about the axis $C\ D$ foreign to the conditions. This effect could be minimised by concentrating the mass as a bob weight, and making the lever and pivot piece carrying it as light as possible. The difficulty could be entirely eliminated by arranging a duplex apparatus, in which two sliding bars are employed, having *opposite rotation*, their preponderating weights being arranged at opposite ends.

§ 228. Dines' Experiments (continued). Aeroplane Experiments. —In the determination of aeroplane data, other than in the special case of the normal plane, the Dines method presents certain difficulties. It may have been noticed from the mechanical disposition of the parts, that it is the *moment of the pressure reaction about the axis $M\ N$* that is the quantity measured, consequently before the magnitude of the pressure reaction can be ascertained the position of the centre of pressure must be known. In order to avoid the necessity of independently determining the centre of pressure, observations are made with the adjustable arm in two complementary positions (Fig. 143 (*a*)), the angle of incidence being the same in both cases. It is evident that the arithmetical mean of the two readings will give the moment of the pressure reaction as if the total force were applied at the geometrical centre.

By investigating two further positions (Fig. 143 (*b*)), an attempt was made to compute the tangential force or skin-friction on the plane: this, however, proved abortive. Mr. Dines attributes the failure of this portion of his investigation to the existence of eddy currents subsequently discovered to have been set up by the frame of the machine.

Experiments with roughened surfaces and with the planes thoroughly wetted showed a diminished reaction, in both cases

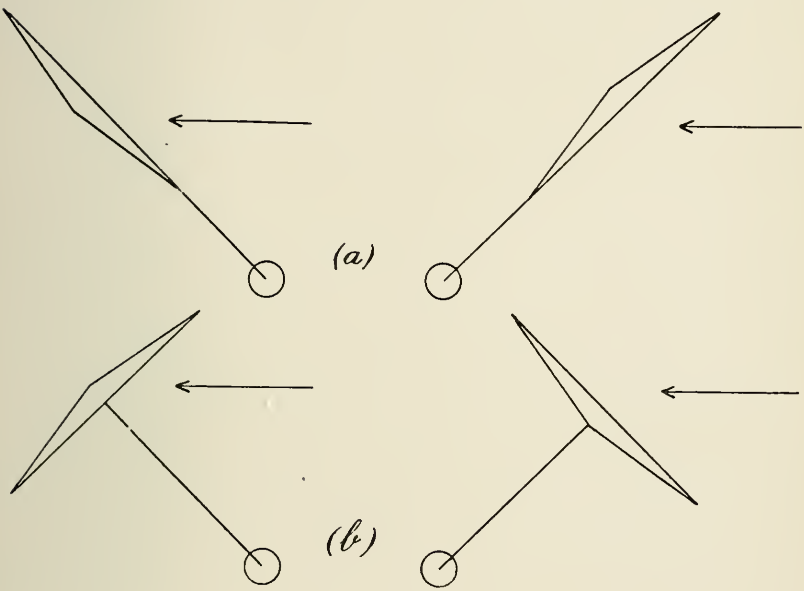


FIG. 143.

equal to about a 20 per cent. drop for the angle of maximum moment when compared with the same plane polished and dry.

Experiments are also recorded, made with the object of determining, in a rough and ready manner, the direction of the stream lines in the immediate vicinity of the surfaces of an inclined plane. A number of pins were driven into the face and back of the plane, and short lengths of coloured silk attached to act as weather-cocks from point to point, and show the local direction of the air currents. The results were drawn from observation; sample diagrams for a square plane at 45 degrees are given, for front and rear aspect, in Fig. 144. It is evident that we have strong evidence here of the *centrifugal shedding of the "dead-water,"* the influence of which has already been the subject of comment in connection with the theory of the screw propeller.

The quantitative results of Mr. Dines' pressure reaction experiments have been given in most part in Chap. VI.

§ 229. Dines' Experiments Discussed.—The simplification resulting from the employment of the *centrifugal balance* in the

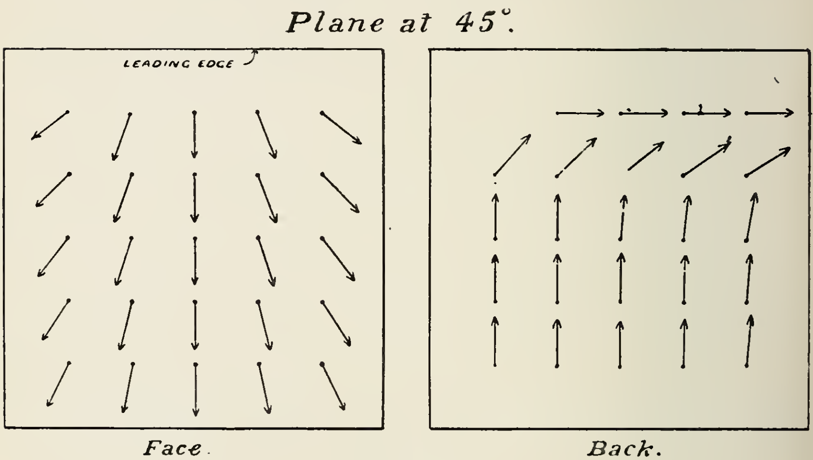


FIG. 144.

apparatus and observations necessary for the determination of data, either of direct resistance or lateral reaction, is very great, and the apparatus in question should certainly be of further service in the future. With a modification such as suggested in § 227, it would appear possible to make determinations with very great rapidity, perhaps even with but a single revolution of the whirler. If this should become possible, one of the chief objections to an indoor whirling apparatus—the residuary motion of the air—would be done away with, and the accuracy of the results greatly increased.

Incidentally the employment of the centrifugal method seems to have demonstrated that the velocity squared law is in all ordinary cases a nearly perfect approximation to the truth.

One of the most remarkable results brought out by these experiments is the peculiar "*kick up*" in the pressure curve (Figs. 93—101, Chap. VI.). This "kick up" seems to have been entirely missed until pointed out by Mr. Dines, although the matter has been investigated by other careful observers. It might be thought that the peculiarity of the Dines curve is related in some way to the triangular section "plane" employed in these experiments, but this hardly seems possible. It would certainly have been more satisfactory if the experiments had been repeated with "planes" of more usual form. For small angles it is highly probable that Dines' results are not accurate, but when once the motion is frankly discontinuous it is difficult to believe that the form of the back of the planes employed can account for so marked a departure of the curve as that observed. It is therefore most probable that the " kick up " is a real feature in the pressure-velocity curve that has escaped the notice of other experimenters.

Dines concludes from his experiments that the effect of skin friction is negligible; one experiment, especially directed as a qualititative test, gave an entirely negative result: no tangential component could be detected. If it were not for the extreme subtlety of the subject it would be difficult to resist the conclusion stated; the pitfalls connected with skin-friction are, however, numerous, and the evidence is inconclusive. It is fair to remark that on this point Dines is in agreement with the late Professor Langley and Sir Hiram Maxim.

§ 230. Langley's Experiments. Method.—The method of experiment adopted by the late Professor Langley resembles that of Mr. Dines in the employment of a whirling table driven by power, and in the use of independent measuring appliances to determine the resistances or reactions on the body subject to investigation.

The following account of these experiments is condensed from Professor Langley's *Memoir*, "Experiments in Aerodynamics," published by the Smithsonian Institute, Washington, 1891.

The site chosen for these experiments was situated in the grounds of the observatory at Allegheny, Pa., U.S.A., some 1,145 ft. above sea level.

The whirling table, erected in the open air, consisted of a trussed cantilever beam, arranged to rotate about a central vertical axis, being driven by an underground horizontal shaft from a separate power house through the medium of bevel gearing. The total length of the beam is given as 60 ft., that is to say, the extremities describe a circle of 30 ft. radius. In construction the beam itself is figured as resembling a light ladder, laid horizontally and stayed from a point about 9 ft. above its centre by a vertical strut, and a number of wire guys taken out to various points along its length. Lateral stiffness is given to the structure by a pair of guys on either side stretched from each extremity to a central outrigger. Provision is made for obtaining at will peripheral speeds from 15 to 100 ft. per second, and for chronographically recording each quarter-revolution by electrical means.

The mode of employment of the whirling table above described involves the use of a number of distinct apparatus, each specially schemed and designed by Prof. Langley for the particular purpose in view.

It is impossible to altogether detach the description of the apparatus from the discussion of its employment and results. In the *Memoir* a chapter is devoted to each instrument, and in the present *précis* and discussion the author has followed a similar arrangement, a separate section being devoted to each chapter of Professor Langley's work.

§ 231. Langley's Experiments. "The Suspended Plane."—This instrument consists of a square plane of thin brass, mounted "slidably" on anti-friction rollers in a frame and suspended by a spring, the frame being mounted on horizontal trunnions. The function of this device is not clear; in the words of the text, its object is "to illustrate an unfamiliar application of a known principle," but the employment of the apparatus does not seem to lead to any results of importance.

§ 232. Langley's Experiments. "The Resultant Pressure Recorder."—This instrument is designed, in the words of the *Memoir*, "for the purpose of obtaining graphically the direction of the total pressure on an inclined plane (in practice a square plane), and roughly measuring its amount." The instrument consists of a beam (Fig. 145) hung symmetrically at its centre in gimbal joints, and carrying at its outer extremity the "wind plane," that is, the plane under investigation, and at its opposite end a tracing point or pencil adapted to record on a sheet of diagram paper, arranged at right angles to the beam itself. A co-ordinate combination of tension springs is employed to hold the beam radial to the whirling table, and the whole is accurately counterpoised so that the plane is virtually weightless; thus, so long as the apparatus is at rest the pencil point is central or at the co-ordinate zero, but when the whirling table is in motion the total reaction on the plane is measured, both as to direction and magnitude, by the resulting displacement of the pencil point. In order to obviate friction the pencil is held away from the recording paper until the desired velocity is reached, when it is released by means of an electro-magnet. Due precautions are taken to ensure proper calibration; to this end the entire spring system is carried in a revoluble frame shown in the figure.

The method of employment is described as follows: "The wind plane is set at an angle of elevation β; a disc of paper is placed upon the recording board and oriented so that a line drawn through its centre to serve as a reference line is exactly vertical. The whirling table is then set in motion, and when a uniform velocity has been attained a current is passed through

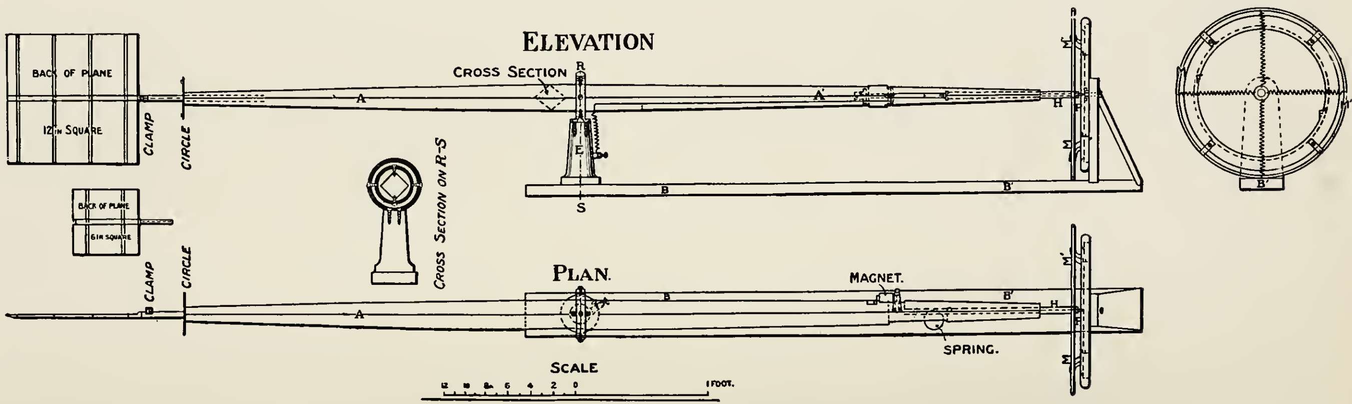

ELEVATION
CROSS SECTION
BACK OF PLANE
12 IN SQUARE
CLAMP
CIRCLE
A
A'
R
S
E
H
F
B
B'
BACK OF PLANE
6 IN SQUARE
CROSS SECTION ON R-S
PLAN.
MAGNET.
SPRING.
SCALE
12
10
8
6
4
2
0
1 FOOT.

Fig. 145.

the electro-magnet, and the pencil records its position on the registering sheet. Since gravity is virtually inoperative on the counterpoised plane, the position of this trace is affected by wind pressure alone. Thus the instrument shows at the same time the direction and magnitude of the resultant wind pressure on the plane and for different velocities of the whirling table.

The results achieved with this instrument were as follows:—

(1) The confirmation of the law of pressure as: $P = k\ V^2$, and the determination of the value of k for the normal plane.

(2) The determination of the *pressure-angle* curve for the square inclined plane, incidentally providing a substantial confirmation of Duchemin's formula.

In addition to the above, Langley claims to disprove " the *assumption made by Newton* that the pressure on the plane varies as the square of the sine of its inclination," and elsewhere he states: "Implicitly contained in the *Principia*, prop. xxxiv., Book II." Now, whatever Langley's experiments prove or disprove, the assumption that he attributes to Sir Isaac Newton is *one that he did not make*, and nothing of the kind is "*implicitly contained*" in the proposition to which reference is made.[1]

Prof. Langley further states that the experiments with this instrument " further show that the effect of the air friction is wholly insensible in such experiments as these." Now, as bearing on this contention, Fig. 1 from the *Memoir* is here reproduced (Fig. 146), and shows the result of plotting a series of observations, with an averaging curve drawn to indicate the probable true values. It will be noted that this curve does not pass through the origin, but cuts the axis of y at a point representing (on the ordinate scale) a matter of some 8 per cent.; or, if we take the possible extremes indicated by the observed points, this quantity will be something between 3 and 13 per cent.

The discrepancy is accounted for by Prof. Langley on p. 24 of

[1] *Vide* "Principia." (The author relies on the translation by Andrew Motte, 1803.)

the *Memoir*, and a "corrected" curve drawn accordingly. The explanatory paragraph is as follows:—

"The values in the Tables are subject to a correction resulting from a flexure of the balance arm and its support. It was

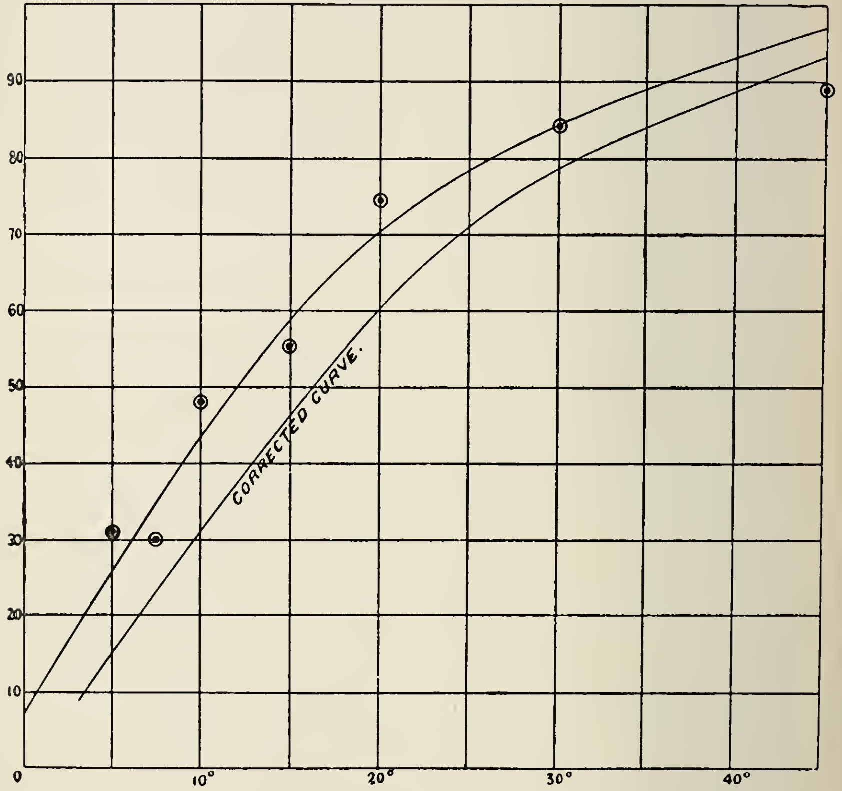


Fig. 146.

observed that the trace of the plane, set at 90 degrees, did not coincide with the horizontal (*i.e.*, the perpendicular to the vertical) line marked on the trace, but was uniformly 4 degrees or 5 degrees below it; so that the angle between the vertical and the trace of the plane did not measure 90 degrees, as had been assumed, but uniformly 94 degrees or 95 degrees, the average being 94·6 degrees. This result was found to be due to the bending backwards of the balance-arm and its support by the pressure of the wind, while the recording board and plumb line presented only a thin edge to the wind, and consequently remained relatively fixed. During motion, therefore, the plane actually had an inclination to the horizon about 5 degrees greater than the angle at which it was set when at rest. This flexure seemed to obtain for all angles of experiment, but with indications of a slightly diminishing effect for the smaller ones; consequently the pressure ratios above given for angles of 45, 30, 20 degrees, etc., really apply to angles of about 50, 35, 25 degrees, etc. After making this correction the final result of the experiments is embodied in the line of Fig. 1 designated *corrected curve*."

Now the author has determined the coefficient of "skin-friction," and it has been shown in the present work that it is nowise a negligible factor; the value would (under the conditions of experiment) in all probability be about 2 per cent., and when the angle of the plane is sufficient to give rise to motion of the discontinuous type it will be in effect about half this amount; the value would require to be considerably less than this before it could be considered as negligible. It is a quantity of this order that Langley confidently asserts does not exist, because it has remained unrecognised in the results of an experiment of which he himself writes as in the paragraph quoted, and which has been subjected to a correction, on very doubtful grounds,[1] many times greater than the quantity involved. The remedy for so serious an error was obviously to redesign the apparatus with a *symmetrical frame*; had this been done there is every probability that the effects of skin-friction would have been clearly recognised.

[1] These words are fully justified. If the correction were required *for the reason stated* then it would be of many times greater magnitude when the plane is normal than when it presents a small angle to the line of flight. Langley says: "This flexure *seemed* to obtain for all angles of experiment, but *with indications of slightly diminishing effect* for the smaller ones."

§ 233. Langley's Experiments. "The Plane Dropper."—The *plane dropper*, as its name implies, is an apparatus in which the aeroplane is allowed to fall under the influence of gravity against the aerodynamic resistance encountered in its flight. In this instrument the "*plane*" is clamped to a "*falling piece*" arranged with friction rollers to slide freely on a vertical guide bar; a detent is employed to hold the falling piece in its top position until released by an electro-magnet. The angle made by the plane to the line of flight has a range of adjustment from horizontal up to 45 degrees. The total fall permitted is four feet, and the time of fall is registered electrically, both at the top and bottom, and later in the experiments at each foot of fall, the observatory chronograph being employed.

The experiments made with the *plane dropper* are numerous, and the results are highly instructive from a qualitative point of view; it would not seem, however, that the method is one that should be imitated by future experimenters: the results are in general deficient in quantitative value, except in the special case when the plane is recorded as "just soaring." The weak point in this kind of instrument is the uncertainty that must prevail as to the existence or otherwise of a *steady state*. During the first portion of the drop there is *acceleration* taking place, that is to say, part of the weight of the paraphernalia is spent in overcoming its own inertia, and only a portion is supported aerodynamically; so that a considerable calculation is necessary before the results recorded can be made quantitatively available.

Admitting its defects, the method is one that appeals strongly to the imagination, imitating as it does many of the conditions of free flight, and in the hands of Prof. Langley it was shown capable of giving some valuable information. The chief points demonstrated were as follows:—

(1) That the time of fall of a horizontal plane in horizontal motion is greater than when no horizontal motion exists, and is greater the greater the horizontal velocity.

(2) The vital importance of the *shape* of the plane and of its *aspect* on the weight supported.

(3) The existence of a critical angle at which the aspect effect undergoes reversal.

(4) The fact that planes can be superposed in flight without sensible diminution of their individual supporting power, provided that they are separated by a certain minimum distance. For planes fifteen inches by four inches in pterygoid aspect the minimum distance between the superposed planes was found to be about four inches, or approximately equal to the "fore-and-aft" dimension.

It is curious that, although Langley in many places elsewhere in the *Memoir* has pointed out the failure of the $\sin^2$ law (the law of the Newtonian *medium*) as applied to air, he apparently overlooks the fact that the falling plane is on this point actually the *experimentum crucis*, for it has been shown (§§ 145—50) that if the $\sin^2$ law holds good in any fluid or medium, the rate of fall of a horizontal plane will be independent of and unaffected by its horizontal motion.

We find once more, in the chapter dealing with the plane dropper, the assumption that skin friction is negligible, resulting in much false inference. This time the error is tacitly assumed. No further proof is announced. The statements that are affected by this error are sufficiently numerous. The following example will put the reader of the *Memoir* on his guard :—

"The results of these two series of experiments furnish all that is needed to completely elucidate the proposition that I first illustrated by the suspended plane, namely, that the effort required to support a bird or flying machine in the air is greatest when it is at rest relatively to the air, and diminishes with the horizontal speed which it attains, and to demonstrate and illustrate the truth of the important statement that *in actual horizontal flight it costs absolutely less power to maintain a high velocity than a low one.*"

Later we find :—

". . . For the former case this is 0·0156 horse-power, and for the latter case approximately 0·0095 horse-power—that is, less power is required to maintain a horizontal velocity of seventeen metres per second than of fourteen, a conclusion which is in accordance with all the other observations and the general fact deducible from them, that it costs less power in this case to maintain a high speed than a low one—a conclusion, it need hardly be said, of the very highest importance, and which will receive later independent confirmation."

"Of subordinate, but still of very great, interest is the fact that if a larger plane have the supporting properties of this model, or if we use a system of planes like the model, less than one-horse power is required both to support in the air a plane or system of planes weighing 100 lbs., and at the same time propel it horizontally at a velocity of nearly forty miles per hour."

§ 234. Langley's Experiments. The "Component Pressure Recorder."—This is by far the most important of the appliances originated by Prof. Langley for use in conjunction with the whirling table, and is one that should receive careful study from future experimenters. In construction, the component pressure recorder somewhat resembles the resultant pressure recorder already described, but instead of measuring the *magnitude* and *direction* of the total reaction by a symmetrical spring combination, the reaction is resolved into its horizontal and vertical components, which are separately recorded, the former directly on a chronograph cylinder forming part of the instrument, and the latter by the condition that the "*soaring speed*" is reached. It may be remarked that, whereas in the resultant pressure recorder the plane is counterpoised so as to be virtually weightless, in the present instrument the weight of the plane, loaded to whatever extent desired, is used as a measure of the vertical component.

The drawings of this instrument as figured in the *Memoir* are

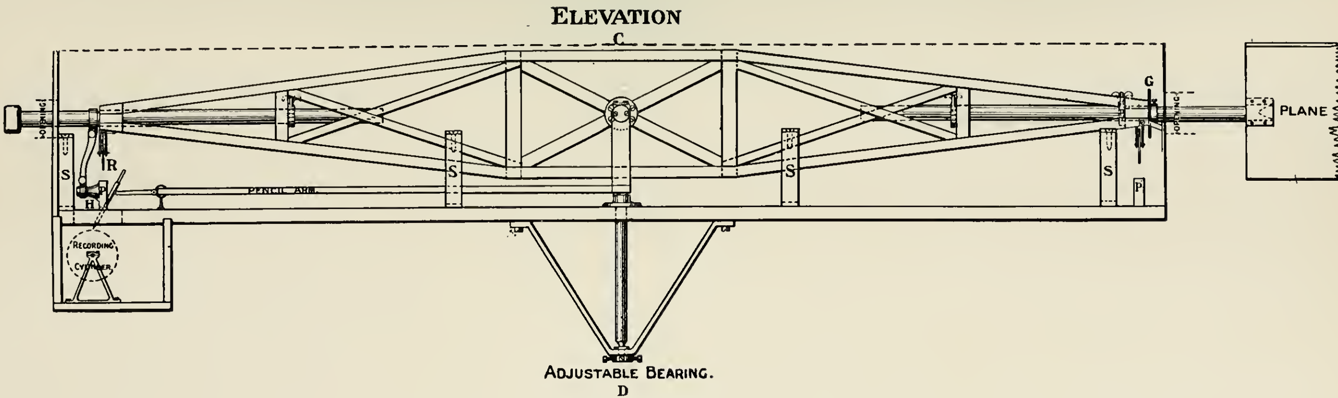

ELEVATION
C
OPENING
S
R
H
PENCIL ARM
RECORDING CYLINDER
S
S
S
G
P
OPENING
PLANE
ADJUSTABLE BEARING.
D

FIG. 147.

given in Fig. 147, in which a light stiff beam, built on the lattice principle, is mounted on knife-edge gimbals on a frame which is in turn free to rotate about a vertical axis. This beam, which is functionally comparable to the beam of a balance, carries at its outer end (*i.e.*, the end remote from the axis of the whirling table) the "wind plane," attached by a tubular arm to the divided circle G, by which it may be set to any desired angle. The inner and outer limbs of the beam are symmetrical, and the whole is enclosed in a box or case to afford shelter from the wind. A dummy end is allowed to project at the inner end to balance (as to wind pressure) the attachment arm at the outer extremity.

A pencil arm (indicated as such in the figure) is provided, attached to the gimbal frame, to record direct on the chronograph drum. This pencil arm also serves to attach the spring by which the horizontal component is measured. Friction wheels R are fitted at both ends of the beam to limit the vertical movement.

The delicacy of suspension was found to be greater than could be employed under outdoor conditions, and a brush H was added to develop a certain regulated amount of friction.[1]

The chief work accomplished with the component pressure recorder was the following:—

(1) The determinations for planes of different proportions, of the velocity of "soaring" corresponding to different values of angle and load. Incidentally, the existence of an *angle of reversal*, already mentioned in connection with the *plane dropper*, was clearly brought out, the previous result being confirmed.

(2) The determination of the values of P_β / P_{90} for planes of different aspect ratio.

(3) The determination by direct measurement of the horizontal component of the reaction on planes at different angles and soaring speeds supporting a known weight.

Fig. 148 gives, plotted to a reduced scale, one of the soaring speed diagrams taken from the *Memoir*. The "reversal" is well

[1] A crude makeshift.

shown in the case of the curves *A* and *B*, whilst the curve *C* shows that for planes of this particular proportion and aspect the curve for small angles approximates very closely to the law $V^2 \times \beta =$ constant. The theoretical curve is given by the dotted line showing the degree of approximation. This law has already been given in the form—$\beta \propto \frac{1}{V^2}$ (§ 152 (*e*)).

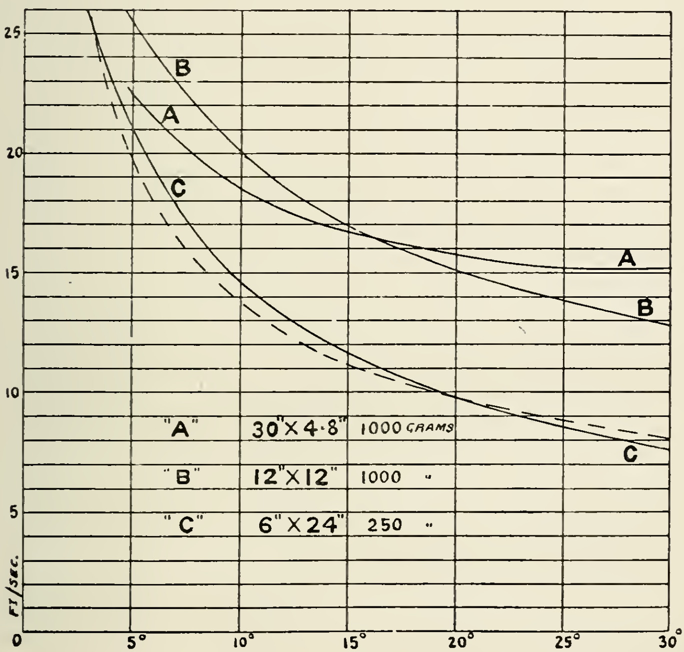


FIG. 148.

Fig. 99 (Chap. VI.) is an example taken from the *Memoir* (Fig. 10), giving the P_β/P_{90} relationship for changes in β value, as determined by means of the present appliance.

Fig. 149 is also a plotting (Fig. 11 of the *Memoir*), showing the direct determination of the horizontal component. The small circles represent the actual observations, and the crosses give points calculated on the basis of a simple resolution of forces, assuming the reaction exactly at right angles to the surfaces of the plane—that is to say, neglecting skin-friction. The curve drawn is a so-called *corrected curve*, the basis of the

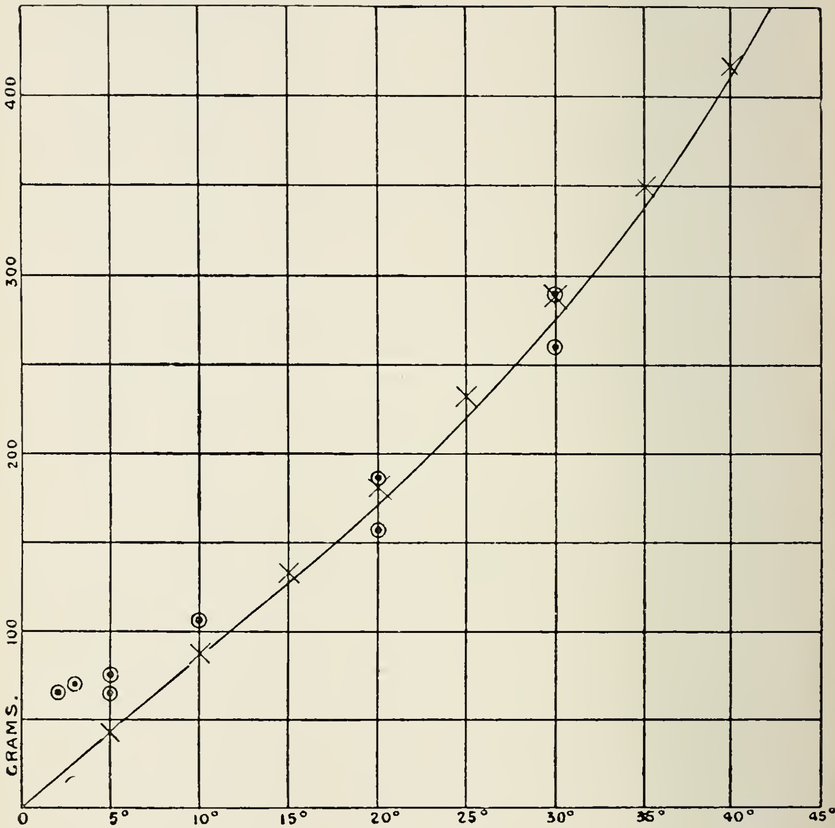


FIG. 149.

correction being given in the *Memoir*, and which we will now proceed to examine.

Let us first draw two curves, from theoretical considerations alone, on the basis (*a*) that the fluid is frictionless, and (*b*) that there is a tangential force due to a coefficient of skin friction $\xi = {\cdot}025$ effective on one face only of the plane—that is, an effective value = ·0125 (see § 182); it is also supposed that the tangential velocity of the air is proportional to *cos* β—that is, equal to V *cos* β. The two curves thus plotted are given in Fig. 150.

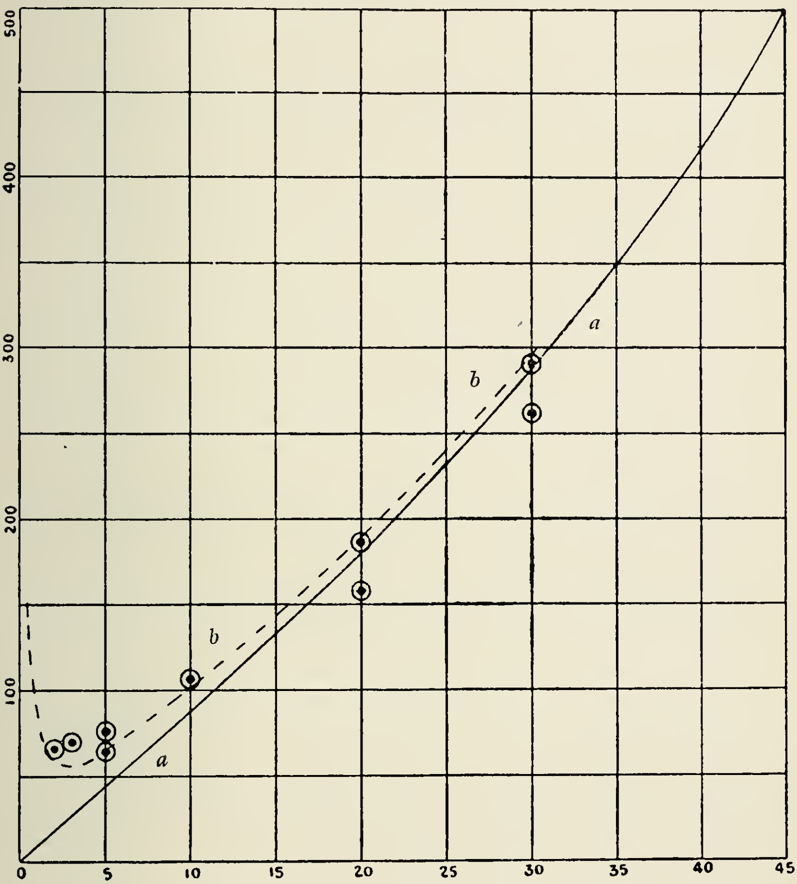


FIG. 150.

Now, it is evident that the curve (*b*) (dotted line) corresponds more closely to the actual observed values than the curve (*a*), and we have certainly here *primâ facie* evidence of a tangential force about equal to that which we have assumed, and which is deduced from the author's own experiments.

Let us see what Langley says on the subject. We find (p. 63) the story of the plotting told in a few words :—

". . . These values have been plotted in Fig. 11, and a smooth curve has been drawn to represent them as a whole. For angles below 10 degrees the curve, however, *instead of following the measured pressure, is directed to the origin, so that the results will show a zero horizontal pressure for a zero angle of inclination.*"

It may be remarked parenthetically that here the *complete assumption* has been made of that which it should have been the *function of the experiment to prove.* The author of the *Memoir* continues :—

"This, of course, must be the case for a plane of no thickness, and cannot be true for any planes of finite thickness with square edges, though it may be and is sensibly so with those whose edges are rounded to a so-called fair form. Now the actual planes of the experiments presented a squarely cut end surface one-eighth of an inch 3·2mm. thick, and for low angles of inclination this end surface is practically normal to the wind. Both the computed pressures for such an area and the actually measured pressures, when the plane is set at 0 degree, indicate conclusively that a large portion of the pressures measured at the soaring speeds of 2 degrees, 3 degrees, and 5 degrees, is *end pressure*, and if this be deducted the remaining pressure agrees well with the position of the curve. The observed pressures, therefore, when these features are understood, become quite consistent. The curve represents the result obtained from these observations for the horizontal pressure on a plane *with 'fair' shaped edges* at soaring speeds."

The above argument appears to the present author to be excusable as an attempt to explain why the results of one experiment or series of experiments might differ from some other experiment or established fact, but it does not constitute a demonstration that skin friction is negligible. The fallacy of an argument on these lines has been already pointed out in

Chap. VI., § 158. It is in any case very difficult to defend plausible reasoning of this kind, *when the actual experiment with planes of "fair" form could have been tried for the expenditure of an additional few shillings and with but little delay.*

The inevitable statement as to the power expended in flight follows :—

"The most important conclusion may be said to be the confirmation of the statement that *to maintain such planes in horizontal flight at high speeds, less power is needed than for low ones.*

"In this connection I may state the fact, surely of extreme interest as bearing on the possibility of mechanical flight, that while an engine developing one horse-power can, as has been shown, transport over 200 pounds at a rate of 20 metres per second (45 miles per hour), such an engine (*i.e.*, engine and boiler) can be actually built to weigh less than one-tenth of this amount."

§ 235. Langley's Experiments. The "Dynamometer Chronograph."—This apparatus was devised for the measurement of the thrust and torque of screw propellers, as a means of practically testing trial models and ascertaining efficiency obtainable. For the details of the instrument reference should be made to the *Memoir.*

In the chapter on the use of this appliance, Prof. Langley explicitly states that the details of his investigations are reserved for future publication ; certain particulars are, however, vouchsafed, including a sample determination, which is of considerable interest in view of the theory of the preceding chapter.

It would appear from the data given that the propeller employed, having a diameter of 30 inches, had an effective pitch of 1 foot approximately, that is to say, its radius was $1\frac{1}{4}$ times the pitch.

Referring to our efficiency diagram (Fig. 127), we see that this denotes the employment of rather more than the whole of the diagram given, so that the efficiency will vary over the length of the blade from 70·4 per cent. to about 40 per cent.; if we take the mean as a rough approximation of the efficiency value to be expected, we have 55 per cent. The actual efficiency obtained was 52 per cent., which is quite as near as could be anticipated.

Again, as to the number of blades, Langley found that two blades gave a better result that any greater number of blades. Now the rule laid down in § 218 can hardly be relied on in the present case: the design of this propeller is abnormal. We may fall back on § 211. In the propeller under discussion the thickness of the peripteral zone (§ 210) will be evidently nearly as great, if not quite as great, as the pitch, consequently we must be approaching the point at which one blade will interfere with itself, and two blades will certainly overlap to some extent. It is consequently quite evident that any increase on the number must be detrimental. Thus we again find substantial confirmation of the peripteral theory. The concluding words of the chapter on the *Dynamometer Chronograph* are singularly to the point in view of the conclusion in § 211 on the comparison in theory of the aerial and marine propeller. Professor Langley says:—

" . . . It may be said that, notwithstanding the great difference between the character of the media, one being a light and very compressible, and the other a heavy and very incompressible, fluid, these observations have indicated that there is a very considerable analogy between the best form of aerial and of marine propeller."

§ 236. Langley's Experiments. The "Counterpoised Eccentric Plane."—An apparatus devised for determining the variations in the positions in the centre of pressure, for varying angles of inclination of a plane to its line of flight.

This appliance follows on established lines, the point of suspension of the plane being fixed for each trial and the angle of equilibrium being experimentally recorded. The result of these experiments has already been given; in the main the work of previous investigators receives confirmation (Fig. 94, Chap. VI., § 148).

§ 237. Langley's Experiments. The "Rolling Carriage."—This instrument is a highly specialised contrivance for investigating the law of pressure on the *normal plane*, and for determining with a greater degree of accuracy than previously the value of the constant relating to same.

The instrument consists of a frame beautifully mounted on friction rollers, and recording direct on a chronograph barrel. The wind plane is attached to the front end of a bar, carried forward from the frame and clamped thereto, the pressure on the wind plane being taken by a carefully calibrated spring and the deflection recorded on the chronograph drum.

The experiments made with this instrument proved disappointing, the results, owing to the open air conditions, being no more consistent than those previously obtained with the *resultant pressure recorder*. The value of C, cited in Chap. VI., is that given by Langley as determined by the *rolling carriage;* the value, however, is substantially the same as that previously ascertained with the earlier instrument.

§ 238. Langley's Experiments. Summary.—Prof. Langley concludes his account in the *Memoir* with a *summary*. Much of this deals with the question of the power required for flight, where naturally the same error is made as elsewhere in the work, the energy necessary to support in a *frictionless fluid* alone being taken into account.

It is from no wish to belittle the work of the late Prof. Langley that attention has so frequently been drawn to the point at issue. Langley's name will always stand as one of the most distinguished pioneers of experimental aerodynamics. The whole of the mis-statements to which attention has been directed hinge upon the one fundamental error, that of the assumption of the negligibility of skin friction; and if the whole *Memoir* be prefaced by the words, "*neglecting the influence of skin-friction*," Langley's position would be substantially regularised.

Professor Langley's work has, however, been widely read, and his statements, unqualified as they stand, have been commonly accepted, and it is therefore impossible in a work of this type to be too emphatic in denouncing the errors in question.

It would seem probable that the publication of the "Experiments in Aerodynamics" was unduly hastened; it would otherwise be difficult to account for the repeated misleading citation of Newton (pp. 4, 8, 15, 24, 25, 89, and 105), when a moment's reference to any reliable edition of the *Principia* would have prevented any such mistake. Newton dealt with a hypothetical medium clearly defined in the enunciation to prop. xxxiv., and not with air at all, and the proposition cited is perfectly sound.

Beyond this the mathematical analysis constituting Appendix "B" is scarcely convincing. Also the calculation forming the second footnote, p. 9, the details of which are not given, is manifestly conducted on insufficient premises. This calculation purports to be a theoretical proof of the negligibility of skin-friction as confirming the supposed experimental result.

So far as the experimental work itself is concerned, apart from inference, it is undoubtedly the most valuable contribution to our knowledge that has so far appeared, with the exception perhaps of the work of Dines already discussed. The general results of Langley's experiments are entirely confirmatory of the theory set forth in the present work, but the experiments suggest that we have in our theory carried the "*small angle*" hypothesis to about its limit, and that if we have to deal with angles greater than those tabulated in Chap. VIII. some correction or refinement of method may become necessary.

§ 239. The Author's Experiments.—The author has investigated experimentally many of the problems connected with aerial flight. The greater part of these investigations relate to the subject matter of the later volume, "Aerodonetics," and only certain experiments having an immediate bearing on the aerodynamics of flight will be dealt with at the present juncture.

A method of experiment that the author has used to some advantage involves the employment of gliding models. Up to the present time the whole question of the stability of such models has been but little understood, and it is necessary to some extent to anticipate the conclusions of the later portion of the work.

It is currently believed that the equilibrium of a bird in flight is essentially maintained by the intervention of the brain and nerve centres, and that an aerodrome or *aerodone*, in order that it should possess stability, must be fitted with parts capable of ready and rapid adjustment, and furnished with some "brain equivalent," or be immediately directed by an aeronaut. It may be stated at once that no such provision is necessary, and that a properly designed rigid structure is capable of maintaining its own equilibrium, and possesses complete stability within prearranged limits; and further, that such a rigid structure (or aerodone) may be designed to automatically simulate many of the apparently life-like movements of birds in flight or at will glide steadily at its natural velocity at a constant angle of flight path.

The above are mere bald statements of fact, that will be fully substantiated in the subsequent volume. The behaviour and stability of an aerodone in flight will for the present be taken for granted, an indication of the underlying principles having been given in § 162 on the *Ballasted Aeroplane*.

§ 240. Scope of Experiments.—The scope of the present series of experiments has been in the main confined to the determination of ξ by a variety of methods.

This quantity, which has been frequently stated to be negligible, is (as has been demonstrated in the present work) one of very great moment in relation to the dynamics of flight, and the determination of ξ with a reasonable degree of accuracy is therefore a matter of prime importance.

Incidentally, data are obtained from which other aerodynamic constants may be deduced, though in this respect the method of free flight experiment has not furnished as reliable data as may be expected in the future when more suitable apparatus has been elaborated.

The various methods employed by the author do not give results that are altogether in accord, but in view of the extent of the general disagreement in the work of other investigators, and in the difficulties of determining ξ in particular, this is in no way surprising.

Experiment apart, there is a *primâ facie* case for the existence of a coefficient skin-friction of some considerable magnitude in the fact that the similar coefficient, as determined by Froude and others, is, in the case of water, a matter of *one per cent.* or thereabouts, and in the fact that the kinematic viscosity of air is *fourteen times as great* as that of water.

§ 241. Author's Experiments. Method.—Three modifications of the free flight method of experiment have been employed; these may be enumerated as follows :—

(1) *The Added Surface Method.*—In this an aerodone is first constructed on the lines laid down in patent specification 17935 of 1905 (Fig. 151), the auxiliary surface being made about the minimum necessary for stability. The natural gliding angle and velocity are very carefully measured from trial "glides." The auxiliary surface is then increased by gumming extension laminæ on to the fins, care being taken not to alter the total weight or the position of the centre of gravity. Further glides are then made and the angle and velocity are again measured; the added resistance is then calculated from the data obtained, and so the value of the ξ is determined.

(2) *The Total Surface Method.*—An aerodone is constructed of

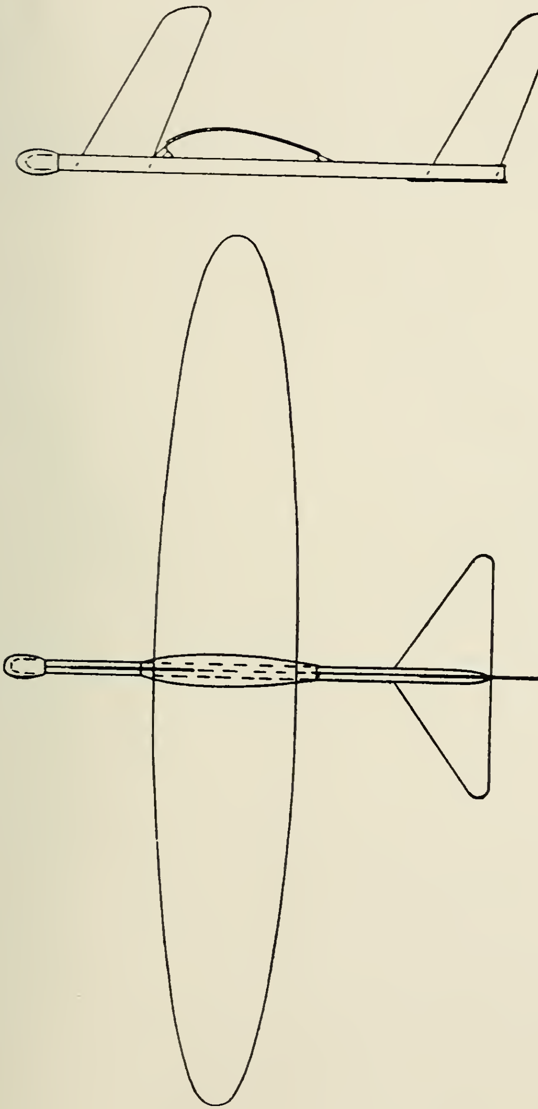


Fig. 151.

special design, Fig. 152, the supporting member being a plane whose centre of pressure is known from independent experiment. The tail plane is divided into two portions arranged so as to be as little as possible affected by the wake disturbance; this is essential on account of the fact that the angle between the supporting plane and the tail plane is assumed to be the angle made by the former by the line of flight. The computation of ξ

Fig. 152.

is made from gliding data, the whole surface being assumed as subject to skin-friction, or an allowance may be made in respect of the supporting plane on the lines laid down in §§ 182, 183 and 184.

(3) *The Method of the Ballasted Aeroplane.*—Reference has already been made to this method (§ 162). A number of planes are prepared of exactly the same size and total weight, but with their centres of gravity situated at different distances from their geometrical centres. Trial flights are made, and the resulting data give simultaneous equations from which the values of the constant may be deduced.

§ 242. Author's Experiments. Method (*continued*).—In addition to the free flight experiments enumerated above, an attempt has been made to effect the direct measurement of ξ by means of a new instrument, which may be appropriately termed an *aerodynamic balance*.[1]

The magnitude of ξ as determined by the method of free flight suggested that, in spite of the failure of previous experimenters, it should be quite possible to effect a direct measurement of this quantity by the aid of a suitably designed appliance.

The aerodynamic balance, Fig. 153, consists of a horizontal arm or beam A, pivoted about a vertical axis B, the amplitude of motion permitted being regulated by the screws CC, which also form electrical contacts.

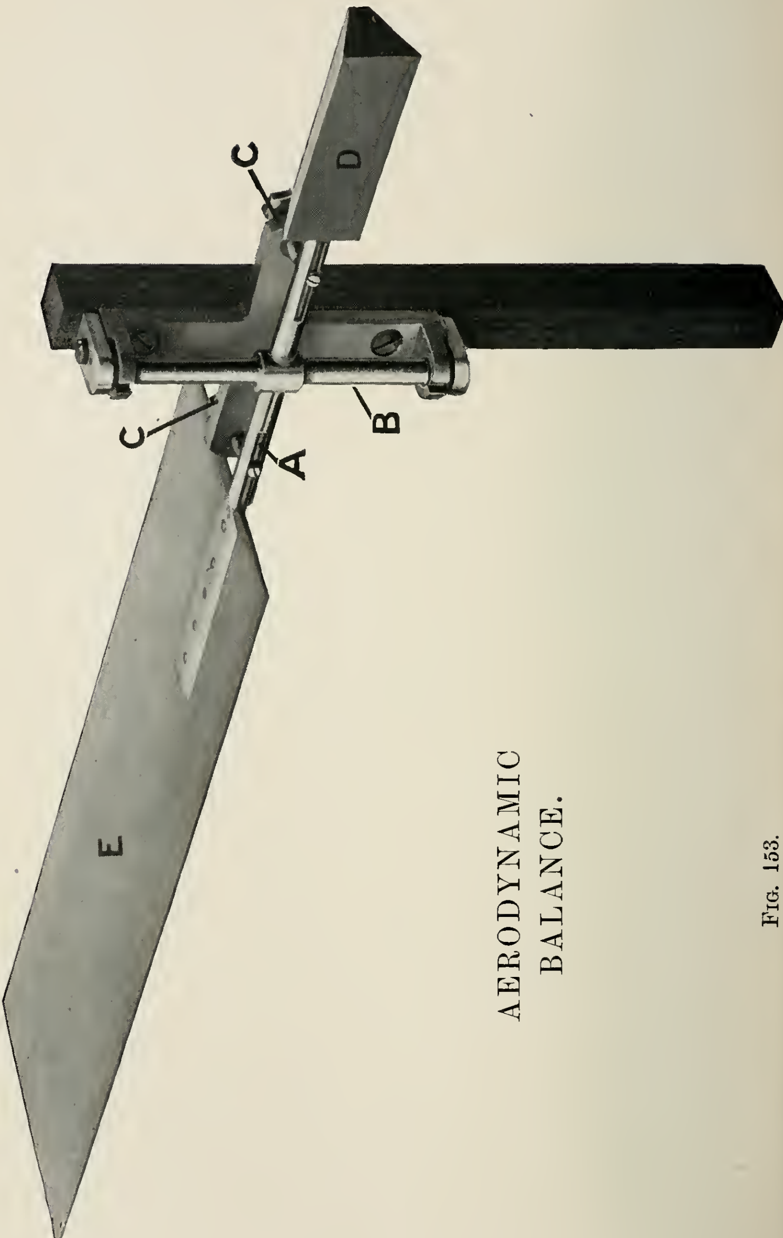
For the determination of ξ, a normal plane D is attached to one end of the beam and a friction plane E to the other, the areas of the two being adjusted until they exactly balance. The instrument is used either by being exposed to the wind and held stationary, or fitted in front of an automobile vehicle in still air.

In either case the planes require to be carefully balanced about the vertical axis so that gravitation and inertia forces are inoperative. When the instrument is held stationary this precaution is unnecessary so long as the axis is exactly vertical, but it is more convenient to have the instrument properly balanced in any case. In spite of every precaution, when the instrument is mounted on a motor car the beam is found to be in a continual state of oscillation between its stops, probably due to slight rotational movements of the car body produced by the unevenness of the road. This difficulty was actually experienced to so great an extent that the employment of the instrument in its present form on a motor vehicle was abandoned.[2]

The uses of the aerodynamic balance obviously are not

[1] A prototype of the aerodynamic balance was employed by Dines (see § 49).

[2] The difficulty could be overcome by re-designing the apparatus with two beams having opposite rotary movement. (Comp. § 227.)



AERODYNAMIC BALANCE.

FIG. 153.

confined to the determination of ξ; the instrument may be used quite generally as a comparator of the resistance of planes of various shapes or of different solid forms.

§ 243. Method of Added Surface.—Mica Aerodone. Series C., No. 1, Fig. 154.

Weight (after adjustment of ballast) = ·60 gram.

Aerofoil, elliptical, $4\frac{1}{2}$ in. × $\frac{3}{4}$ in.; actual area	= 2·65 sq. in.
Tail plane, area	= ·50 ,, ,,
Back-bone, surface ÷ 2 = equivalent area .	= ·48 ,, ,,
Fin area (without added surface) . . .	= ·14 ,, ,,
Total area (without added surface) . . .	= 3·77 ,, ,,
Added surface	1·06 ,, ,,

Sept., 1905. *Trial of model without added surface.*

LAUNCHED FROM 7 FT. ALTITUDE.

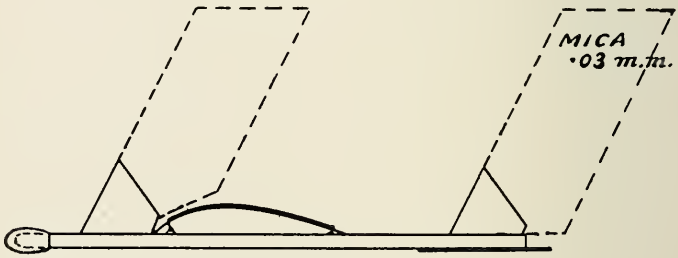
	Distance. Ft.	Time.[1] Secs.	Velocity. Ft./Secs.
1	37	3·2	11·55
2	40	3·2	12·50
3	34	3·2	10·62
4	35	3·0	11·66
5	37	3·2	11·55
6	32	2·8	11·43
7	36	3·2	11·25
Total . .	251	21·8	80·56
Mean . .	35·8	3·11	11·50

Whence, $\gamma = \dfrac{7}{35{\cdot}8} = {\cdot}1955$ or resistance in line of flight = ·1955 × ·60 = ·1172 grams.

[1] Taken by stop watch.

Sept., 1905. *Trial of model, with added surface.*

	Distance. Ft.	Time. Secs.	Velocity. Ft./Secs.
1	29	2·6	11·15
2	29	2·6	11·15
3	32	2·8	11·44
4	28	2·8	10·00
5	34	?	
6	33	2·8	11·80
7	31	2·6	11·80
8	36	2·8	12·84
Total . .	8) 252	7) 19·0	80·18
Mean . .	31·5	2·71	11·45



AERODONE. SERIES C. No 1.

FIG. 154.

Whence, $\gamma = \frac{7}{31·5} = ·222$ or resistance in line of flight = ·222 × ·60 = ·1332 grams.

∴ Resistance due to added surface = ·1332 − ·1172 = ·016 grams.

∴ resistance per sq. ft. added surface $= \frac{·016 \times 144}{1·06} =$ 2·175 grams.

Now pressure per sq. ft. in pounds at 11·5 ft./sec. is given by expression

$$P = \frac{\cdot 7 \rho V^2}{g} = \frac{\cdot 7 \times \cdot 078 \times 11\cdot 5 \times 11\cdot 5}{32\cdot 2} = \cdot 224$$

which in grams becomes

$$\cdot 224 \times 453\cdot 6 = 101\cdot 5 \text{ grams.}$$

$$\therefore \quad \xi = \frac{2\cdot 175}{101\cdot 5} = \cdot 0214$$

The above example is one of several determinations made by this method. Generally speaking, the flight measurements showed greater variation than in the example given; the day of these experiments was exceptionally calm, and the aerodone used (No. 1) made a long series of good straight glides without mishap; a performance which it is not always easy to obtain. Flights of circular or otherwise curved path need to be rejected.

The results of different series of experiments were found to give values of ξ varying from a trifle over ·012 to nearly ·030 as a maximum.

Using the value above determined ($\xi = \cdot 0214$) we may calculate the total skin resistance of the model employed.

Total area (without added surface)

= 3·77 sq. in. = ·0262 sq. ft. or

resistance = ·0262 × 101·5 × ·021 = ·0558 grams.

But total resistance = ·1172 grams, hence we may audit the resistance account for this model as follows:—

Frictional	= ·0558
Aerodynamic	= ·0614
Total .	= ·1172

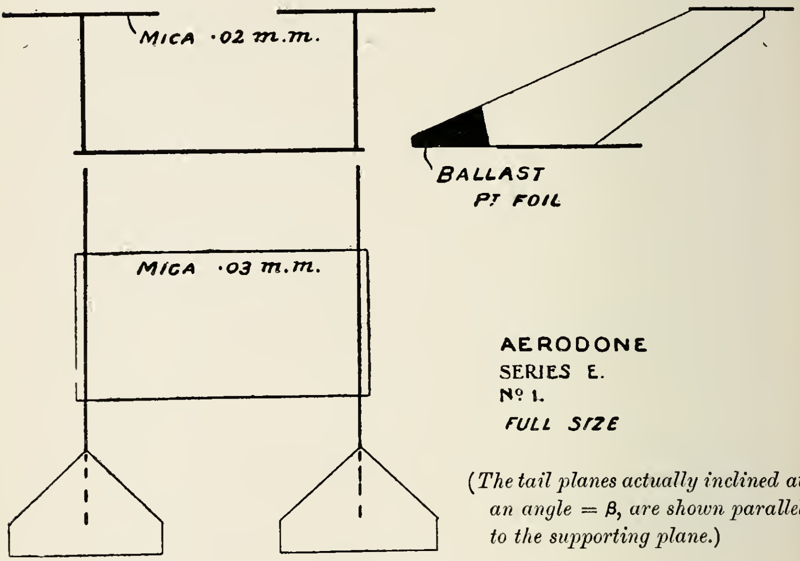
A result which appears to be quite consistent, as showing the model to be approximately designed for the conditions of least resistance within the limits of experimental error.[1] (§ 164.)

[1] We may compare the pressure relation of this model with the values laid down in Table IX., § 185.

Area (effective) of aerofoil on $\frac{2}{3}\frac{L^2}{n}$ basis is

§ 244. Method of Total Surface.—Mica Aerodone. Series E., No. I. Fig. 155.

This aerodone was one of a series of models specially designed for the purpose of measuring ξ by the method of total surface. The supporting member is an aeroplane whose angle to the line



(*The tail planes actually inclined at an angle = β, are shown parallel to the supporting plane.*)

Fig. 155.

of flight is determined by a pair of tail planes whose sole function is *directive*. It may be noted that the tail planes are so placed as to be influenced equally by the downward and upward

---

$$\frac{20{\cdot}25 \times 2}{6 \times 3} = 2{\cdot}25 \text{ sq. in.} = {\cdot}0156 \text{ sq. ft.}$$

or weight carried $= \dfrac{{\cdot}6}{{\cdot}0156}$ grams per sq. ft. $= 38{\cdot}5$ or $\dfrac{38{\cdot}5 \times 32{\cdot}2}{453{\cdot}6} = 2{\cdot}73$ poundals.

Now $V = 11{\cdot}5$ or $V^2 = 132$ or $P/V^2 = \dfrac{2{\cdot}73}{1{\cdot}32} = {\cdot}0214$ against ${\cdot}0286$ given in Table IX.

This difference is not more than might be expected in view of the present state of knowledge.

components of the terminal vortices, so that no error shall be introduced by an allowance for the "downthrow current," as would be necessary were the tail plane situated in the ordinary position (Fig. 151).

Weight ·24 grams.
Area 1·125
+ 1·030
+ ·490
= 2·645 sq. in.
Angle of aeroplane $\frac{1}{20}$.

Flight Data.

	Distance. Ft.	Time. Secs.	
1	30	2	Launched from 10 ft. altitude.
2	33	2	
3	28	1·8	
4	21	?	Launched from 7 ft. altitude.

Whence,
Mean $V = 15{\cdot}5$ ft./sec.
Mean $\tan \gamma = \frac{1}{3}$ or $\gamma = 18\frac{1}{2}°$ or,
Resistance $= \sin 18\frac{1}{2}° \times {\cdot}24$ . . . . . . $= {\cdot}075$
Aerodynamic resistance $= W\beta = {\cdot}24 \times {\cdot}05$ . . . $= {\cdot}012$

∴ Skin resistance . . . . . . $= {\cdot}063$

And area = 2·645 sq. in. or skin resistance per sq. ft. area—

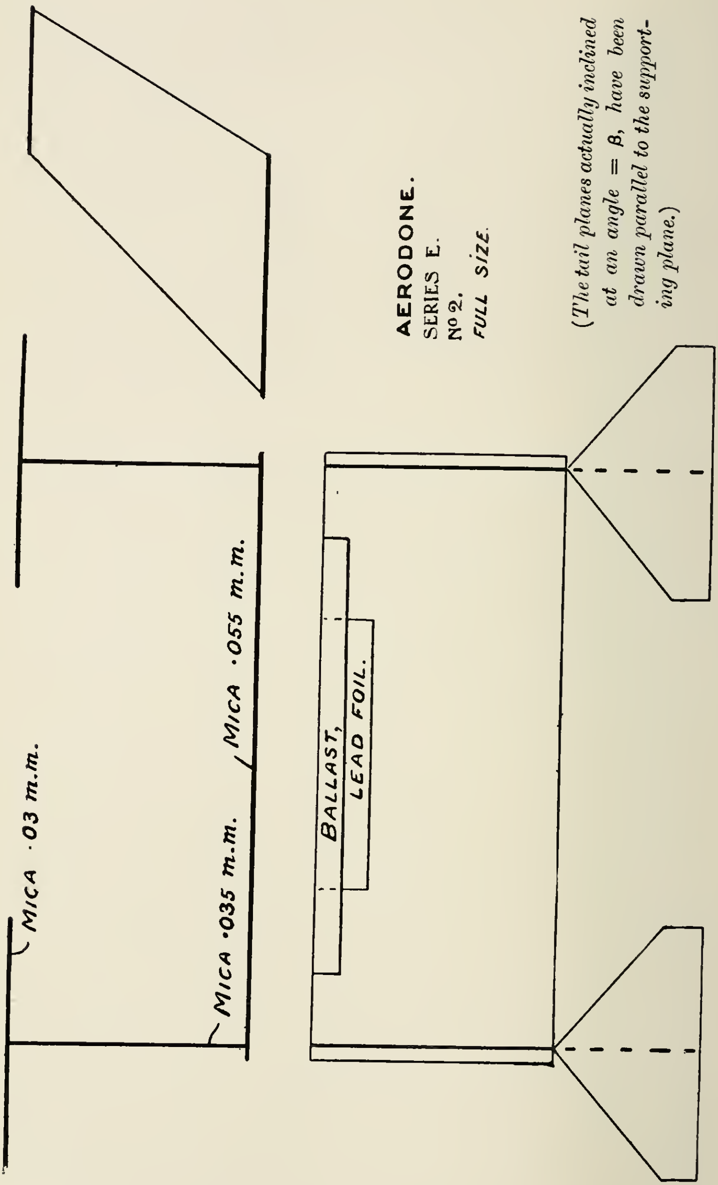
$$= \frac{{\cdot}063 \times 144}{2{\cdot}645} = 3{\cdot}43 \text{ (grams)}.$$

$$\frac{3{\cdot}43 \times 32{\cdot}2}{453{\cdot}6} = {\cdot}243 \text{ (poundals)}.$$

But normal plane reaction at 15·5 ft./sec.

$$= {\cdot}7 \times {\cdot}078 \times 15{\cdot}5 \times 15{\cdot}5 = 13{\cdot}1$$

$$\therefore \quad \xi = \frac{{\cdot}243}{13{\cdot}1} = {\cdot}0185$$



(*The tail planes actually inclined at an angle = β, have been drawn parallel to the supporting plane.*)

FIG. 156.

A further trial with this model, repaired after damage, gave a value of $\xi = \cdot025$. The cause of this divergence was not ascertained.

Model No. 2. *Series E.* (Fig. 156).—Trials with this model gave the result ξ = about ·03. This value is probably too high. The weather was unfavourable, and the weight of the model (1·66 grams) proved too great for the method of construction; frequent repairs had to be made in the course of a single series of experiments.

Model No. 3. *Series E.* (Fig. 152), *Construction.*—Aeroplane and fins, varnished cedar. Tail planes, mica plates. Body, cedar, ballasted with lead.

Weight = 46·3 grams = ·102 lbs.

Area—

Aeroplane	44
Tail plane	13
Fins .	12·4
Body .	3·25
	72·65 sq. in.
	= ·504 sq. ft.

Angle of aeroplane (β), $= \frac{1}{20}$.

Preliminary Trial. Aerodone launched by hand, 13 ft. 6 in. altitude.

Flight Data.[1]

	Distance. Ft.	Time. Secs.	Velocity. Ft./Sec.
1	39	1·4	28
2	42	1·2	35
3	40	1·4	28
4	42	1·2	35
5	42	1·2	35
	5) 205	6·4	161
Mean	41	1·3	32

[1] Flights of curved or irregular path were not recorded. Time taken with stop watch reading to ·2 second.

Whence $\tan\gamma = \frac{13{\cdot}5}{41} = {\cdot}33$ or $\gamma = 18\frac{1}{4}^\circ$ $\sin\gamma = {\cdot}313$.

Resistance in line of flight

$$= W\sin\gamma = {\cdot}102 \times {\cdot}313 = {\cdot}032 \text{ lbs.}$$

Aerodynamic resistance

$$= W\beta = \frac{{\cdot}102}{20} \qquad = {\cdot}005 \;\;,,$$

$\therefore$ Skin resistance . . . $= {\cdot}027$

or per sq. ft. $= \frac{{\cdot}027}{{\cdot}504} = {\cdot}0536$ (pounds)

$= 1{\cdot}725$ poundals.

But the normal plane reaction,

$$P = {\cdot}7 \times {\cdot}078 \times 32 \times 32 = 55{\cdot}8 \text{ (poundals)}$$

or $$\xi = \frac{1{\cdot}725}{55{\cdot}8} = {\cdot}0309$$

It was evident in the course of the above trial that sufficient velocity was not being given to the aerodone, that is to say, its projected velocity was less than its natural velocity and that the necessary velocity could not be given by hand throwing without sacrificing accuracy.[1] A further series of trials made with a catapult launching device gave data as follows :—

Aerodone launched from 20 feet altitude.

	Distance. Ft.	Time. Secs.	Velocity. Ft./Sec.
1	66	1·8	36·6
2	62	1·8	33·2
3	68	1·8	37·8
4	Model collided with tree and damaged.		
	3) 196		107·6
Mean	65	1·8	36

[1] Any rotational movement imparted at the moment of projection is most detrimental to the gliding path.

Whence, $V = 36$ ft./sec. $\tan \gamma = \cdot 246$ or $\gamma = 13° - 50'$ or $\sin \gamma = \cdot 239$.

Resistance in line of flight

$$= W \sin \gamma = \cdot 102 \times \cdot 239 = \cdot 0244$$

Aerodynamic resistance

$$= W\beta = \frac{\cdot 102}{20} = \cdot 0050$$

$\therefore$ Skin resistance (lbs.) . . . $= \cdot 0194$

or in poundals per sq. ft.

$$= \frac{\cdot 0194 \times 32 \cdot 2}{\cdot 504} = 1 \cdot 24$$

But normal plane reaction at 36 ft./sec. (poundals),

$$P = \cdot 7 \times \cdot 078 \times 36 \times 36 = 71$$

$$\therefore \quad \xi = \frac{1 \cdot 24}{71} = \cdot 0174$$

If we make an allowance in respect of the aeroplane in accordance with §§ 182, 183, and 184, deducting half the area, we have, 72·65 − 22 = 50·65 sq. in. = ·351 sq. ft. in lieu of ·504 as above. Or skin resistance in poundals per sq. ft.

$$= \frac{\cdot 0194 \times 32 \cdot 2}{\cdot 351} = 1 \cdot 78$$

$$\therefore \quad \xi = \frac{1 \cdot 78}{71} = \cdot 025$$

In the case of an aerodone having a natural velocity as high as 36 ft./sec., it is impossible to be sure, in so short a flight as 65 feet, that the true natural velocity and gliding angle are recorded ; in a short flight the launching velocity and angle have a serious influence on the flight path.

If the altitude of discharge could be increased to 50 feet or thereabouts, with a flight path of some 200 feet length, this difficulty would be overcome, or at least its importance would be reduced to a negligible quantity.[1]

[1] The author had intended repeating these experiments under more favourable circumstances, but the difficulty of hitting the right weather conditions, at an appointed place, away from home, at a time that is otherwise convenient, has hitherto proved insuperable.

§ 245. The Method of the Ballasted Aeroplane.—The method of the ballasted aeroplane not only permits of the determination of the coefficient of skin-friction but simultaneously provides data from which the constant *c* and the relation between the angle and centre of pressure of the aeroplane may be calculated.

The following examples will serve for the purposes of illustration. A standard form of aeroplane has been employed throughout (Fig. 109), measuring 8 in. by 2 in., and ballasted by a lead shot presenting a resistance taken as equivalent to ·025 sq. in. of normal plane. The weights of different planes employed for any given series of experiments are all brought up to the same amount by gumming lead-foil in the region of the centre of gravity, the only difference between the different planes of a series being the position of the centre of gravity, and therefore the position of the centre of pressure, and consequently the angle of equilibrium.

The launching of the planes was in all cases effected by means of a launching stick, the aeroplane being placed on a small platten on the top of a straight stick, the lower end of which is held about shoulder high, the act of launching being accomplished by swaying the body so as to give an approximately parallel motion. A certain degree of skill is easily acquired, and a reasonable percentage of good straight flights may be obtained without difficulty.

Example,

Two planes, weight $5\frac{1}{4}$ grams (·372 poundals).

Launching data.

	Velocity.	7·5 ft. Altitude.
No. 1	15 ft./sec.	47 ft. mean glide.
No. 2	12·5 ft./sec.	35 ft. ,, ,,

$$\text{Or, } V_1 = 15 \text{ ft./sec.}$$
$$V_2 = 12{\cdot}5 \text{ ft./sec.}$$
$$\gamma_1 = \frac{7{\cdot}5}{47} = {\cdot}160 \quad [1]$$
$$\gamma_2 = \frac{7{\cdot}5}{35} = {\cdot}214$$
$$x_1 + y_1 = {\cdot}160 \times 5{\cdot}25 = {\cdot}84 \qquad (1)$$
$$x_2 + y_2 = {\cdot}214 \times 5{\cdot}25 = 1{\cdot}12 \qquad (2)$$

Now we know that $x = n\ V^2$ and $y = \frac{m}{V^2}$ where n and m are constants,

$$\text{or, } x_1 = 15^2 n, \quad y_1 = \frac{m}{15^2} \qquad (3)$$
$$x_2 = 12{\cdot}5^2 n, \quad y_2 = \frac{m}{12{\cdot}5^2} \qquad (4)$$
$$\text{By (1) and (3)} \quad 15^2 n + \frac{m}{15^2} = {\cdot}84 \qquad (5)$$
$$\text{,, (2) ,, (4)} \quad 12{\cdot}5^2 n + \frac{m}{12{\cdot}5^2} = 1{\cdot}12 \qquad (6)$$
$$\therefore \quad 12{\cdot}5^2 n + \frac{({\cdot}84 - 15^2 n) \times 15^2}{12{\cdot}5^2} = 1{\cdot}12$$
$$\therefore \quad 156\ n - 325\ n = 1{\cdot}12 - 1{\cdot}21$$

or,
$$169\ n = {\cdot}09$$
$$n = {\cdot}000532$$

or,
$$x = {\cdot}000532\ V^2 \text{ grams.}$$

multiplying by $\frac{144}{16}$ to obtain grams *per square foot*, and by $\frac{32{\cdot}2}{453{\cdot}6}$ to convert to British absolute units, we have—

$$V^2 \times {\cdot}000532 \times \frac{144}{16} \times \frac{32{\cdot}2}{453{\cdot}6} = {\cdot}00034\ V^2$$

But normal plane pressure is given by expression—

$$P_{90} = C\ \rho\ V^2$$
$$\text{or} \quad \xi = \frac{{\cdot}00034}{C\ \rho} = \frac{{\cdot}00034}{{\cdot}7 \times {\cdot}078} = {\cdot}0062$$

In the foregoing calculation no allowance has been made for

[1] In these experiments the hypothesis of the *small* angle is taken as applying to γ values; γ is expressed in radians.

the direct resistance of the ballast. Taking this as the equivalent of a normal plane area of ·025 sq. in., or ·000174 sq. ft., and multiplying by $\frac{144}{16}$ to bring up to a *per sq. ft.* basis, we have ·00156, or resistance per sq. ft. due to ballast

$$= \cdot 00156\ C\ \rho\ V^2$$

$$\text{or } \xi = \frac{\cdot 00034 - \cdot 00156\ C\ \rho}{C\ \rho}$$

$$= \frac{\cdot 00034}{7 \times \cdot 078} - \cdot 00156 = \cdot 0062 - \cdot 0015$$

$$\xi = \cdot 0047$$

Again,

Two planes, weight 5·9 grams (·418 poundals).

	C.g. distance from front edge.	Velocity.	8 ft. altitude.
No. 3	25%	17 ft./sec.	54 ft.
No. 4	30%	13 ft./sec.	37·5 ft.

or,

$$V_3 = 17 \text{ ft./sec.}$$

$$V_4 = 13 \text{ ft./sec.}$$

$$\gamma_3 = \frac{8}{54} = \cdot 148$$

$$\gamma_4 = \frac{8}{37 \cdot 5} = \cdot 213$$

$$x_3 + y_3 = \cdot 148 \times 5 \cdot 9 = \cdot 875 \qquad (1)$$

$$x_4 + y_4 = \cdot 213 \times 5 \cdot 9 = 1 \cdot 26 \qquad (2)$$

$$x_3 = 17^2\, n, \quad y_3 = \frac{m}{17^2} \qquad (3)$$

$$x_4 = 13^2\, n, \quad y_4 = \frac{m}{13^{\cdot 2}} \qquad (4)$$

By (1) and (3) $$17^2\, n + \frac{m}{17^2} = \cdot 875 \qquad (5)$$

,, (2) ,, (4) $$13^2\, n + \frac{m}{13^2} = 1 \cdot 26 \qquad (6)$$

$$\therefore \quad 13^2 n + \frac{(\cdot 875 - 17^2 n) \times 17^2}{13^2} = 1\cdot 26$$

$$\therefore \quad 169\, n - 494\, n = 1\cdot 26 - 1\cdot 495$$

or, $$325\, n = \cdot 235$$

$$n = \cdot 000724$$

or, $$x = \cdot 000724\ V^2 \text{ grams.}$$

or, in poundals per square foot

$$\cdot 000724 \times \frac{144}{16} \times \frac{32\cdot 2}{453\cdot 6} = \cdot 000461$$

$$\xi = \frac{\cdot 000461}{\cdot 7 \times \cdot 078} - \cdot 0015$$

$$= \cdot 0085 - \cdot 0015 = \cdot 007$$

the deduction ·0015 being made, as in the last example, for ballast resistance.

Determination of constant c.

$$W = A\ P_\beta = A\ c\ \beta\ P_{90} = A\ c\ \beta\ C\ \rho\ V^2$$

or, $$\beta = \frac{W}{A\ c\ C\ \rho\ V^2}$$

but, $$y = W\beta \quad \therefore \quad y = \frac{W^2}{A\ c\ C\ \rho\ V^2}$$

and, $$y = \frac{m}{V^2}$$

$\therefore$ $$m = \frac{W^2}{A\ c\ C\ \rho}$$

or, $$c = \frac{W^2}{A\ m \times \cdot 0546}$$

all quantities in absolute units.

Thus, for the determination of c in any particular case the value of m must first be obtained from the equations, the remaining quantities in the expression A and W being the area (sq. ft.) and weight (poundals) of the aeroplanes employed.

Example.—Planes 1 and 2.

Flight data as given.

By (5) $$m = 15^2\ (\cdot 84 - 15^2\ n)$$

where $$n = \cdot 000532$$

or, $$m = (225 \times \cdot 84) - (225 \times \cdot 1197)$$
$$= 189 - 26 \cdot 9$$
$$= 162 \cdot 1$$

This is the value of m for y expressed in *grams;* for y in *poundals* this becomes—

$$m = 11 \cdot 5$$

and for the aeroplanes in question,

$$A = \cdot 111 \text{ and } W = \cdot 372$$

$$\therefore \quad c = \frac{\cdot 372^2}{\cdot 111 \times 11 \cdot 5 \times \cdot 0546} = 1 \cdot 98$$

This is about the value as determined directly by Duchemin, Dines and Langley for the square plane; it is probably too low for a plane of $\mathbf{n} = 4$ as used in these experiments.

Example.—Planes 3 and 4.

$$m = 17^2 \times (\cdot 875 - 17^2\, n)$$

where $$n = \cdot 000724$$

whence $$m = 192 \cdot 5$$

or, when absolute units are employed, $m = 13 \cdot 6$.

$$A = \cdot 111 \quad W = \cdot 418.$$

$$\therefore \quad c = \frac{\cdot 418 \times \cdot 418}{\cdot 111 \times 13 \cdot 6 \times \cdot 0546} = 2 \cdot 11$$

a result which is still probably less than the true value.

Calculation of β.

$$\beta = \frac{y}{W} = \frac{m}{W\, V^2}$$

Taking planes 3 and 4.

Plane No. 3.

$$\beta = \frac{192 \cdot 5}{5 \cdot 9 \times 289} = \cdot 112 \qquad \beta^\circ = 6 \cdot 5^\circ$$

or,

Plane No. 4.

$$\beta = \frac{192 \cdot 5}{5 \cdot 9 \times 169} = \cdot 193 \qquad \beta^\circ = 11 \cdot 1$$

Plotting these results on the basis of § 148 for the known positions of centre of pressure for these planes, we have diagram Fig. 157.

The divergence shown in the above determinations is largely due to the temporary and insufficient character of the apparatus employed, and to the fact that for want of suitable accommodation the experiments were conducted out of doors.

It is further possible that the considerations raised in §§ 182, 183, to some extent invalidate the present method. So long as the type of fluid motion in the periptery of the aeroplane is frankly discontinuous the method will in theory give consistent results, but so soon as the live stream touches the upper surface

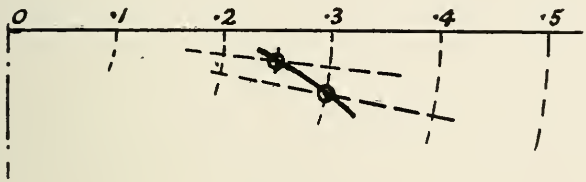


FIG. 157.

of the plane, as it must do when the angle becomes very small, the area subject to skin-friction will increase in some way as an inverse function of the angle, and the equation $x = n V^2$ will cease to hold good. We may consequently anticipate that when the angle β becomes less than some critical value the curve will cease to be of the form plotted in Fig. 112, and the present method will break down. It is principally for this reason that the author has confined his observations to the low velocity portion of the curve; it will be time enough to carry these observations further when better launching and measuring appliances have been developed.

§ 246. Determination of ξ by the Aerodynamic Balance.—In the determination of ξ by the aerodynamic balance, one arm of the beam A, Fig. 153, is furnished with a lead block D, Fig. 158, whose sectional form is an isosceles triangle, the base of which is formed by the face in presentation, being the normal plane the pressure on which constitutes a measure of the reaction on

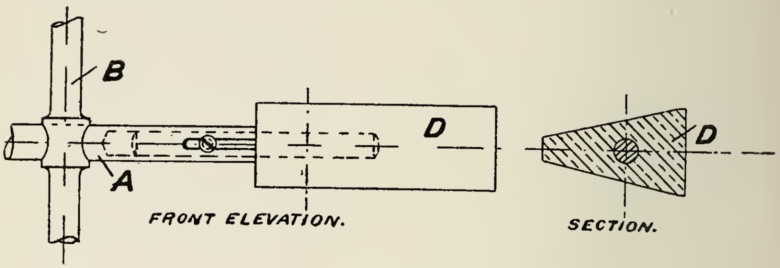


FIG. 158.

the friction plane. The area of the normal plane can be extended at will by cementing a mica plate to its face, the edges of which are clipped until exact balance is obtained.

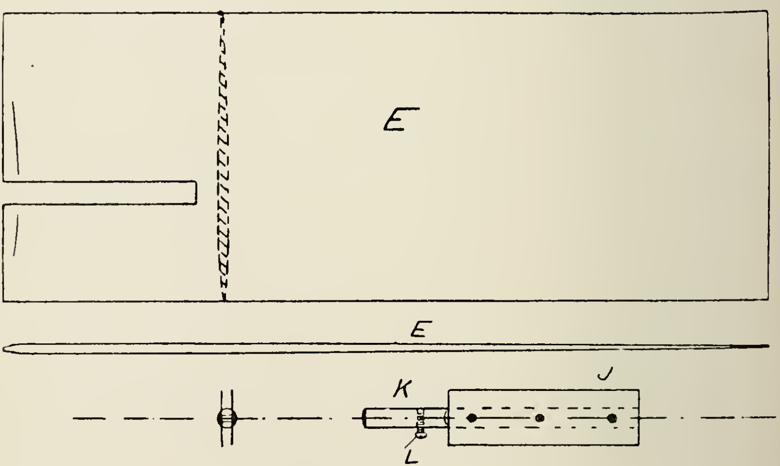


FIG. 159.

The opposite arm of the beam carries the friction plane E (Fig. 159); this is carefully formed of cedar of 6 mm. maximum thickness, the grain being well filled and served with a thin coat of varnish, or otherwise finished as may be required. The friction plane is carried on a holder formed by two pen steel plates J riveted to a shank K which fits into a socket in the balance arm A (Fig. 158) being secured by a set screw L.

In making any determination the area of the normal plane is adjusted until the beam is in equilibrium. The coefficient of skin-friction ξ is then calculated from the relation of the areas of the normal and friction planes multiplied by their respective distances from the pivot axis.

Determination, June 19th, 1907, Cobley Hill, Alvechurch.

Wind velocity (estimated) 20 to 40 miles per hour.[1]

Friction plane No. 1, cedar shellac varnished and roughly polished. In pterygoid aspect.

	Length.		Breadth.		Leverage.
Normal plane	2·5″	×	·95″	×	3·25 = 7·7
Friction plane	16·25″	×	5″	×	10·12 = 823

$$\xi = \frac{7 \cdot 7}{823} = \cdot 0936$$

Determination, June 23rd, 1907.

High wind.

Friction plane No. 2, cedar filled and water gilt and burnished. In pterygoid aspect.

	Length.		Breadth.		Leverage.
Normal plane	2·25″	×	·9″	×	3·25 = 7·3
Friction plane	16·25″	×	5″	×	10·12 = 823

$$\xi = \frac{7 \cdot 3}{823} = \cdot 0888$$

Friction plane No. 1 (polished cedar), substituted for No. 2 as above, showed no appreciable change of balance.

With width of normal plane increased to 1 in., both planes, Nos. 1 and 2, gave insufficient reaction to balance pressure on normal plane. It is therefore to be concluded that for a well varnished surface or for polished metal, under the conditions of

[1] Determinations made with aerodynamic balance are approximately independent of velocity of wind; a rough estimate is sufficient for the purposes of record.

experiment, the effective value of ξ is approximately ·09, with a probable error of less than 10 per cent. *plus* or *minus*.

Roughened surfaces. June 23rd, 1907 (later).

Wind, as before.

Friction plane, covered Oakey's No. $2\frac{1}{2}$ glass paper.

	Length.		Breadth.		Leverage.
Normal plane	2·5″	×	1″	×	3·25 = 8·1
Friction plane	12·25″	×	4·6″	×	8·12 = 458

$$\xi = \frac{8·1}{458} = ·0177$$

On the face of it the method of the aerodynamic balance is so direct and straightforward as to leave no possibility of doubt as to the validity of the results. Under constant wind conditions it seems that a change of 10 per cent. can be readily detected, and it appears, therefore, fair to assume that this is the outside limit of experimental error.

If it had been found possible to conduct experiments in *still air* with the instrument in motion, it would have been difficult to resist the above conclusions; it is, however, by no means certain that, under the conditions of wind reaction experiment, the matter is quite as simple as it appears.

There seems to be some possibility that the normal plane and friction plane are not equally affected by the wind turbulence; it is even uncertain whether the influence of turbulence is *in the same direction* in the two cases. We know the effect of turbulence in the case of the normal plane is probably to *increase* the pressure reaction (§ 131), but it is by no means established that the effect is the same in the case of skin-friction. If the direction of motion of turbulence were confined to the line of motion of the main translation, it would certainly seem that the influence of turbulence would be to increase the frictional drag. If, as is actually the case, the motion of turbulence have a component at right angles to the friction plane, it is conceivable that it will give rise to discontinuity in the system of flow, which, on the principles discussed in §§ 182, 183, and 184, may actually result in a *diminution in the tangential reaction.*

§ 247. Author's Experiments. Summary.—The experiments described in the preceding sections are quite convincing from a *qualitative* point of view, although *quantitatively* speaking the results are inconclusive.

Any and all of the methods described should be capable of giving results of a reasonable degree of accuracy—far more so than at present achieved—and the results so obtained should be in closer accord, one with another, than the author has so far been able to demonstrate.

The deficiency in the present experiments is chiefly that of apparatus and opportunity. The launching of free flight models requires a suitable apparatus to be designed, by which the initial velocity shall be placed under definite control; beyond this it must be considered quite essential, if reliable results are required, that experiments should be conducted inside a building; the absolute calm necessary for aerodynamic determinations is so rare a phenomenon as to render outdoor experiment almost impossible. It is only those who have watched and waited for a really calm day who can fully appreciate its rarity. In repeating these experiments it would be well to arrange for the use of a large hall, well secured against draughts; the equilibrium of low velocity models, such as it is necessary to employ, is very sensitive; even the previous motion of a person across the line of flight will affect the gliding path. High velocity models, although possessing greater stability, are not well suited to the determination of aerodynamic data.

The author's conclusions as to the value of ξ have been given in § 157. Some of the experiments here recorded have been made since these conclusions were formulated, but the differences are not such as to render revision necessary. In brief, it would appear that under all practical conditions the coefficient of skin-friction lies between the values ·01 and

·03, rarely being less than the former or greater than the latter.

It is, perhaps, of some interest to record the fact that for air in motion in a pipe the accepted resistance coefficient gives, on the present basis of computation (*i.e.*, for a double surface in terms of the pressure on a normal plane of equal area), a value of $\xi = \cdot 016$, which is in substantial agreement with the present conclusions, in spite of the totally different conditions that obtain.

The author considers that the method of the ballasted aeroplane has not at present had a fair chance of showing its capabilities. The method is one that demands considerable nicety of manipulation. In the absence of any mechanical launching device, it is quite easy to obtain faulty data if any but straight uniform glides are recorded, and if such data are utilised it is as likely as not the values of the constants deduced will be wide of the mark, even negative values being sometimes obtained. The method is one of which the advantages have only very recently impressed themselves on the author, and time and opportunity have been at present lacking to carry out more than a few rough preliminary experiments. The present publication, in this respect, must therefore be regarded more in the light of an exposition of method than a serious experimental demonstration.

GLOSSARY.

Aerofoil (Author), from the Greek ἀέρος and φυλλον, lit. an *air-leaf.* Denoting the organ of sustentation of an aerodone or aerodrome, or the spread wings of a bird. A supporting member (or members collectively) of undefined form; thus *pterygoid aerofoil,* an aerofoil of wing-like form; *plane aerofoil,* an aeroplane, etc., §§ 112, 128, 172.

Aerodone (Author), from the Greek ἀερο-δόνητος, lit. *tossed in mid air; soaring.* To denote a gliding or soaring model or machine; in particular, any gliding or soaring appliance destitute of propelling apparatus or auxiliary parts; in contradistinction to *aerodrome.*

Aerodonetics (Author, see *aerodone*). The science specifically involved in problems connected with the stability or equilibrium of an aerodone or aerodrome, or of birds in flight, and with the phenomenon of soaring. Equivalent to *Aerodromics,* as proposed by Langley (p. vi. footnote 1).

Aerodrome (Langley), from the Greek ἀερο-δρόμος, lit. *traversing the air; an air runner;* originally proposed to denote a gliding or soaring model or machine, or a flying machine of any kind. *Restricted* by the author to the latter signification; a fully-developed flying appliance; a power-propelled *aerodone,* or an aerodone furnished with directive apparatus. *Something more than simple aerodone.* (Preface, p. v., footnote.)

Aerodromics (Langley; see *aerodrome*), originally proposed to denote the science concerned in the equilibrium, etc., of an aerodrome; equivalent to *aerodonetics* as used by the author. Proposed to be *extended* by the author to include the *aerodynamics* and *aerodonetics* of flight. The whole science concerned with the flight of an aerodrome. Thus the present work may be entitled a *Treatise on Aerodromics*, and the whole subject of aerial flight would be dealt with at a College or University by a lecturer or professor of *Aerodromics*.

APTEROID (Author), from the Greek *α, πτερόν* and *ειδος*, the converse of *pterygoid*. Thus *apteroid aspect*, with the greater dimension arranged in the direction of flight; (the reverse to that which obtains in the wing plan-form of birds), §§ 150, 151.

ASPECT (Dict.), proposed by Langley in its present usage to denote the arrangement of the plan-form of an aeroplane, or other aerofoil, in relation to the direction of flight, § 144.

ICHTHYOID (Dict.), fish-shaped, here applied to denote a body of practical stream-line form, § 9.

PERIPTERAL. See *Periptery*.

PERIPTEROID. See *Periptery*.

PERIPTERY (Dict.), proposed by the author in its present usage as denoting the region round about the wing or in the vicinity of the aerofoil (Greek, *περι* and *πτερόν*), § 107. Hence *peripteral*, as in *peripteral theory* (Ch. 4), *peripteral area*, § 210; *peripteral zone*, § 210; *peripteral motion*, § 126 (*see also* footnote 2, p. viii., Preface). Hence also *peripteroid* motion, § 122 (Greek, *περι, πτερόν* and *ειδος*), the form of flow proper to the inviscid fluid in a doubly connected region, resulting from the superposition of a cyclic motion on one of translation. Resembling the *motion in the periptery*, lit. *round-about-the-wing-like*.

PTERYGOID (Dict.), *wing like*. Hence *pterygoid aspect*, with the lesser dimension in the direction of flight, as in the wing plan-form of a bird, § 152.

SWEEP (Dict.), proposed by the author in its present usage to denote the cross-sectional area of the stratum of fluid, supposed by hypothesis to be that to whose inertia the supporting reaction is due, § 160.

APPENDIX I.

INFLUENCE OF COMPRESSIBILITY OF AIR ON THE ENERGY EXPENDED IN FLIGHT.[1]

THE influence of compressibility as affecting the expenditure of energy in flight is best computed from the velocity of wave motion—sound.

The whole theory of Chapter VIII., based on the hypothesis of constant sweep, relates, strictly speaking, as set forth, to the incompressible fluid; it will be shown that the effects of compressibility can be dealt with as a correction, or rather by a preliminary correction, to the figures involved.

Let us write U for the velocity of sound, and, as before, let V be the velocity of flight. Then it is evident that any disturbance will travel forward relatively to the body in flight less rapidly than it will travel backward in the opposite direction in the relation $\frac{U - V}{U + V}$, as in the case of "Döppler's principle." Now, regarding the fluid motion as due to a *field of force* (Chapter IV., § 113), we have the communication of upward momentum diminished, and the communication of downward momentum increased, in like proportion.

Thus in the ideal case of Chapter IV., if we have to deal with a compressible fluid, an expenditure of power becomes necessary

[1] The method here given is founded on a suggestion made by the author in his paper to the Birmingham Natural History and Philosophical Society in 1894. Owing to repeated rearrangements and revisions, it was accidentally omitted from the MS. of the present work.

in accordance with a regime $\epsilon_1 = \frac{U - V}{U + V}$. (1)

(Compare § 172 *et seq.*)

In the extreme case when V becomes equal to U no disturbance can precede the aerofoil in its flight, and the whole reaction will be due to the communication of downward momentum; the cyclic component in the peripteral system vanishes. In the above expression when $V = U$, $\epsilon_1 =$ zero, which leads to the same conclusion.

Let us take the ϵ of Chapter VIII. to be the ϵ proper to an incompressible fluid, and let the symbol employed above, ϵ_1, be the corresponding value when U is the velocity of sound. Then from the foregoing reasoning we have—

$$\epsilon_1 = \frac{U - V}{U + V}\,\epsilon. \qquad (2)$$

This expression is in harmony with equation (1), which relates to the special case where $\epsilon =$ unity.

Example.—Dealing with the highest result tabulated, *i.e.*, 80 ft. sec., and taking $U = 1120$,

$$\epsilon_1 = \frac{1120 - 80}{1120 + 80}\,\epsilon = \frac{13}{15}\,\epsilon\,;$$

that is to say, *for the velocity stated* the value of ϵ employed in Chapter VIII. is too high in the relation 15/13.

But it is evident from the whole argument of Chapter VIII. that the constant ϵ is not the only one of the constants involved in the equations affected by the compressibility of air. In fact, from the reasoning employed (§§ 161, 172 *et seq.*) it would appear that the constants c and κ will also be affected, and we may fairly make the assumption[1] that the constants will remain related in accordance with the equation of § 177, and that consequently we may regard the influence of compressibility as *degrading* the effective value of n from its actual value to a less value in the proportion required by equation (2).

Thus in the case under discussion, if we have to deal with an

[1] The method is evidently no more than an approximation.

aerofoil whose n = 12, we find by Table IV. ϵ = ·75. Taking 13/15ths of this, we have ϵ = ·65, which the table shows corresponds to n = 7. That is to say (assuming the accuracy of these "plausible values"), for a speed of flight of 80 feet a second the corrected values for an aerofoil of aspect ratio n = 12 can be read from the various tables by taking the equivalent aspect ratio n = 7.

APPENDIX II.

A NOTE ON THE COMMUNICATION OF MOMENTUM AND ON THE VELOCITY AND MOMENTUM OF SOUND.

The "Principle of No Momentum," enunciated as a proposition in § 5 of the present work, constitutes so far as the author is aware an innovation in the treatment of problems in fluid dynamics.

The proof of this proposition, indeed the principle itself, is so perfectly simple and obvious, that it is not without some hesitation that it is put forward as new. The consideration of the following examples, involving the simple application of the principle, and leading to results which certainly are not generally recognised, would seem to leave no doubt as to the fact.

Example 1.—The *Vortex Atom Theory* of Kelvin gives considerable trouble in the light of the Principle of No Momentum.

If the fluid be supposed incompressible and of uniform density in its parts, and if we suppose for example a single vortex ring in motion in a rigidly bounded region,[1] it manifestly cannot carry momentum (§ 5), and equally the momentum of a number of such rings must be zero. It is of course possible that such a ring or number of rings may raise the peripheral pressure of the region, that is, the pressure on the walls of the enclosure, but the case of an incompressible fluid and a rigid enclosure is in this respect an indeterminate problem. Thus if, still regarding

[1] The mixed nature of the conception of vortex atoms in a non-atomic enclosure is possibly responsible for the difficulty pointed out. The enclosure, to carry a vortex atom theory to its logical conclusion, should itself consist of an entanglement of vortex rings or filaments.

the fluid as incompressible, we suppose the enclosure to possess some degree of elasticity so as to exert on the fluid a pressure sufficient to prevent cavitation, then the peripheral pressure will undergo no change in consequence of the vortices, for a change of pressure on the walls of an elastic enclosure must be accompanied by a compression or dilatation of the fluid contents. Under these conditions the greater the energy of the vortex system set up in the fluid the lower will become the pressure in the internal part of the region, so that the plus and minus momentum of the equal and opposite flow taking place across any imaginary barrier plane is accounted for by the ordinary static pressure on the confines of the region, and does not give rise to any added pressure.[1]

If we suppose the enclosure rigid and the fluid elastic, the change of pressure due to the vortices on the boundary walls depends upon the *law of elasticity*, and is not a function of the magnitude or energy of the vortex system alone.

The result of the above reasoning is not at all in harmony with accepted views as to the behaviour of vortices as expounded in the Vortex Atom theory.[2] According to the highest authorities the individual vortices carry momentum just as if they were bodies of greater density than the fluid that contains

[1] The author has heard it argued that every stream of fluid passing any imaginary barrier plane carries momentum across that plane, and therefore must result in added pressure between the fluid and the enclosure. Such an argument is evidently unsound; on the fluid tension hypothesis (§ 82) we may regard these internal motions of the fluid as giving rise to tension across the barrier plane, and this tension is equal and opposite to the momentum per unit time transmitted by every current and counter current set up in the fluid, and on the principles discussed in §§ 81, 82, and 83; this applies not only for the whole region, but individually for every small element of the fluid cut by the imaginary plane. Interpreting in the usual way, we see that it is the ordinary hydrostatic pressure on the walls of the enclosure that supplies the necessary force to balance the momentum transferred per second, and that a diminution of pressure in the vicinity of the barrier arises automatically, precisely equivalent to the momentum transference taking place.

[2] *Nature*, xxiv., p. 47, also "Motion of Vortex Rings," J. J. Thomson.

them, instead of being, according to hypothesis, composed of the fluid itself and therefore of the same density. It is not possible in the present work to go fully into the cause of this discrepancy which the author believes to be due to the mathematical theory regarding the vortex ring, as the result of an impulse distributed evenly over the disc area, instead of as the resultant of two equal and opposite impulses, the one applied over the disc area and the other to the confines of the fluid region. This is merely thrown out as a suggestion, but whatever the explanation may be, the case of a vortex ring travelling to and fro in a rigidly bounded region filled with incompressible fluid and *carrying momentum* is presented for consideration to the exponents of the Vortex Atom Theory as involving a flagrant violation of the third law of motion.

Example 2.—*Momentum of Sound Waves.*—This is a question that has been widely discussed of recent years, and one on which different authorities are not altogether in agreement.[1]

If we take it as essential by definition that the passage of a complete wave or train of waves results in no permanent displacement of the particles of the fluid, that is to say, that each particle of the fluid occupies after the passage of the wave train the same position as before its passage,[2] it immediately follows that the mean density of the wave train is equal to that of the undisturbed fluid.[3]

It is therefore evident (as in § 5) that if such a wave train be supposed to travel to and fro in a box (Fig. 160), from end to end, being repeatedly reflected, no movement of the mass centre of

[1] Compare Poynting, Presidential Address, Physical Society, February 10th, 1905, with Rayleigh, *Phil. Mag.*, vol. x., pp. 364, 374, September, 1905.

[2] If this condition is infringed, the motion is obviously not pure wave motion, but comprises a superposed translation.

[3] This is evident, for if A B C be three equidistant points on the line of propagation, the fluid in the regions A B and B C will be identically the same when the wave train has passed from the region A B into the region B C.

the fluid *within the box*, *i.e.*, relatively to the box, can take place, and hence such a wave train possesses *no momentum.*

It follows that if the wave train have an excess of compression or rarefaction so that its mean density is greater or less than that of the undisturbed fluid (the condition of the particles returning to their initial positions being departed from), momentum will be carried positive or negative, as the case may be, exactly as represented by the excess or deficit of density in the wave train.

Thus the momentum carried by any sound wave is a measure of and is measured by the displacement of matter by that sound wave, and if the displacement is zero the momentum is zero.[1]

The question of momentum carried by wave motion is fre-

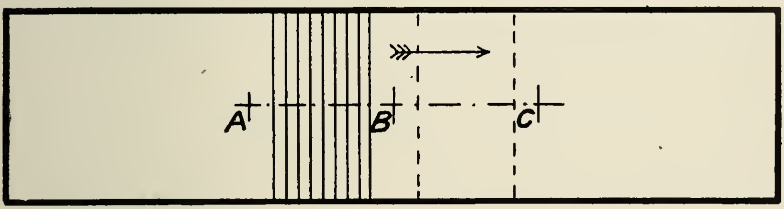


Fig. 160.

quently regarded from the point of view of pressure developed, that is, the pressure produced in the fluid by the communication of momentum at reflection etc. This point of view is not without interest.

Taking first the case of a gas obeying Boyle's law, *i.e.*, $\frac{P}{\rho} =$ constant; the *mean pressure of the whole of the space* can undergo no change, for P/ρ is constant for each small element throughout the region, and the integration of ρ being constant (since the whole mass of fluid in the enclosure is unchanged), the integration of P throughout the enclosure is also unchanged.

[1] There is some want of harmony between this result and the conclusions of many eminent authorities, see Larmor, *Encycl. Brit.*, xxxii., p. 121 b; Rayleigh, *Phil. Mag.*, vol. iii., p. 338, 1902; and Poynting, *l.c. ante.* Rayleigh has amended his conclusions somewhat in a subsequent communication, *l.c. ante.*

Now it by no means follows that the mean pressure *throughout the region* is the same as the mean pressure *on the walls of the enclosure;* in fact, we know from hydrodynamic principles that in many cases of fluid motion it is not so. In the case in point, however, it is manifest that the mean pressure is the same whether the integration is taken over the surface or throughout the volume, for (Fig. 160) the pressure on the walls of the box is point for point the same as for any surface parallel to these walls passing longitudinally through the region, and the pressure on the ends is of the same mean value, for the velocity of sound can be correctly computed on this basis.[1] It is therefore evident that for a fluid obeying Boyle's law the existence of wave motion does not give rise to any change of pressure.

Under these circumstances it follows that change of pressure will take place in a region containing an ordinary gas (P/ρ^γ = constant), the magnitude of which can be calculated from the energy of wave motion that passes into, and exists in, the thermodynamic system.[2]

[1] See Addendum A.

[2] If heat be added to a quantity of a perfect gas contained within an enclosure, the consequent rise of pressure is due to the quantity of heat added and is independent of its distribution. When wave motion exists in such a gas, heat is abstracted where the gas is rarefied and added where the gas is compressed, but more heat is added than subtracted; the difference *represents* the work done, according to well-known thermodynamic principles. We can therefore look upon the adiabatic wave as a Boyle's law wave in which heat has been added to one part and abstracted from another part, but in sum an addition of heat has been made to the contents of the enclosure, and the mean pressure increase can be calculated therefrom.

The fact that the *distribution* of added heat within a vessel does not affect the pressure increase has been taken advantage of by the author (1894) in the construction of an air calorimeter, a small quantity of gas whose calorific value is to be determined being burnt in a large vessel and the rise of pressure noted (see Addendum C). For mechanical reasons the appliance was not a success.

The result that the pressure due to an adiabatic wave can be deduced from the energy entering into the thermodynamic system appears to have been reached independently by Lord Rayleigh.

There is much confusion of thought at the present time on the question of the *carrying of momentum* by a wave train and the *generation of pressure* in a fluid region occupied by wave motion. This has probably arisen from too close attention being paid to the special case of a continuous wave train, as in the Kundt's tube.

It is much more difficult to distinguish between direct momentum transference by the wave and momentum transference by the pressure generated by the wave, in the case of the Kundt's tube, where the whole region is occupied by wave motion than in the case of a limited wave train passing to and fro.

Thus in the case of a limited wave train, if it carry momentum, that momentum can be represented by some definite value of mv, and the remainder of the system with which the wave is associated must, relatively to the common mass centre, have an equal and opposite momentum at every instant of time. But a self-contained system consisting of a simple enclosure containing fluid of uniform mean density (regarding the individual waves of the train as small) cannot suffer change of momentum without infringing the third law of motion; consequently the wave train (if of the same mean density as the quiescent fluid) cannot carry momentum. This is in effect the argument of § 5.

In the case of the continuous train, as in the Kundt's tube, we lose touch with this method of argument, for the action is continuous, and a pressure increase can only be distinguished from the true carrying of momentum by the wave train by a process of mathematical analysis that is full of pitfalls.[1]

The case of light pressure, or the carrying of momentum by electro-magnetic radiation, is not a problem in ordinary dynamics, and is untouched by a purely dynamical argument or method of demonstration such as here employed. The reason for this fundamental distinction is that when motions

[1] Poynting, Pres. Add., Phys. Soc., 1905, p. 397.

of the all-pervading ether are *essentially* involved, such a term as a self-contained system ceases to have any signification. There is only one self-contained system known to us—the Universe.

From another point of view we know that according to modern theory the momentum of any finite quantity of matter, *however small*, moving with the velocity of light is infinite[1]; consequently a finite quantity of momentum will be carried at this velocity by a quantity of matter smaller than can be expressed in finite units, or, physically speaking, communication of momentum at the velocity of light becomes independent of the displacement or transference of matter. Thus the present application of the principle of no momentum is in no way antagonistic to modern views and discovery as to the transference of momentum by light and other manifestations of electric radiation.

ADDENDUM A.

Assuming Boyle's law, let us examine the case of an isolated compression wave travelling to and fro in a prismatic box of unit cross section and length $= l$. Let the mass of the fluid in this wave, *in excess of the normal contents of the region it occupies*, be m.

Now since the wave carries an excess of fluid it will carry momentum, and this momentum will be represented by the mass m transported with the velocity of wave propagation, which we denote by the symbol U.

And the presence of this excess of fluid in the enclosure will raise the mean pressure throughout the enclosure to the same extent as if it were uniformly diffused. (This has already been demonstrated.)

The proposition is to show that the pressure increase due to

[1] J. J. Thomson, "Electricity and Matter," Ch. II., p. 44.

the wave on the ends of the enclosure is equal to the mean pressure increase throughout the enclosure.

The proof of the proposition rests in showing that the velocity of sound can be correctly calculated by the assumption of the proposition as hypothesis.

Thus the compression wave will be in equilibrium when its rate of communication of momentum to the ends of the enclosure is equal to the added pressure.

Momentum of wave . . . . . $= mU$.

„ communicated by each reflexion $= 2mU$.

Number of reflexions per second . . . $= \frac{U}{2\,l}$

∴ force due to momentum of wave . . $= \frac{m\,U^2}{l}$ (1)

Let P_1 be the mean pressure exerted by the additional mass m distributed throughout the enclosure; then, by Boyle's law,

$\frac{P_1}{\rho} = k$, where k is a constant,

and $$\rho = \frac{m}{l}$$

or $$P_1 = \frac{mk}{l} \qquad (2)$$

by (1) and (2) $$\frac{m\,U^2}{l} = \frac{mk}{l}$$

or $$U^2 = k$$

∴ $$U = \sqrt{k} = \sqrt{\frac{P}{\rho}}$$

which is the well-known result; by substituting for P and ρ (abs. units) for air at any stated temperature the *Boyle's law velocity* is obtained; this is, of course, subject to Laplace's correction for the actual velocity.

The above reasoning, though here given as a disproof of Larmor's theorem as a generalisation, is in reality a valid and simple method of determining the velocity of sound. If we cast aside the mythology introduced into the subject by the light radiation specialists, and treat the question as it should be treated, as a matter of ordinary dynamics, it is evident that it *is* the displacement of matter (if any) that gives rise to the momentum of a wave, and it *is* the momentum of the wave that gives rise to the pressure at reflection, and by equating the two, as has been done in the foregoing demonstration, we have the simplest known method of obtaining the expression for the velocity of sound.

The nature of the flaw in Larmor's theorem is discussed in Addendum B of the present Appendix.

The simplicity of the present method of the determination of the velocity of sound is largely due to the form in which Boyle's law is presented. It is usual to write the isothermal law (Boyle's law), for a perfect gas $PV =$ constant; now this presumes *mass constant*. It would be quite as correct to write $P/m =$ constant, taking the volume to remain unchanged. It is obviously best to include both mass and volume as variables and write $P/\rho =$ constant, as has been done.

The present method has much in its favour. The argument not only covers waves of small amplitude, but waves of any amplitude and any form; we may regard a wave in a fluid obeying Boyle's law as built up of a number of superposed elements, each of which conforms to the pressure-momentum equation giving the same value of U for each element alone or in superposition. Consequently waves in a fluid obeying Boyle's law have no tendency to travel faster in one part than in another part; their form is permanent and velocity uniform.

In Poynting and Thomson's "Sound," a method is given for the theoretical determination of the velocity of sound, on the assumption that the pressure changes are proportional to the volume changes, and the usual well-known expression $U = \sqrt{\frac{E}{\rho}}$ is obtained. A foot-note is given in connection with this demonstration, as follows:—

"If the pressure changes are too considerable to justify the "assumption that they are proportional to the volume changes, "we may regard the variation from proportionality as an external "force represented by X. Thus in a wave of very considerable "displacement and pressure excess, $X/(P_M - P_N)$ can be shown "to be positive, and U is greater than the value in (5). This "agrees with certain experimental results given below."

The suggestion here appears to be that the straight line trace in the PV diagram (which is the equivalent of the Poynting and Thomson hypothesis) is essential to the rigid application of theory for waves of sensible magnitude. This is contrary to the result here obtained, and surely must be incorrect. A gas obeying Boyle's law according to these authorities would share with the real gas the mutability of wave form consequent on the adiabatic law.

According to the present author the *straight line* diagram is to be found in the plotting of P and ρ for Boyle's law, Fig. 161 a, which corresponds to the hyperbola for the PV diagram; and this straight line diagram is the looked-for analogue of the isochronous pendulum.

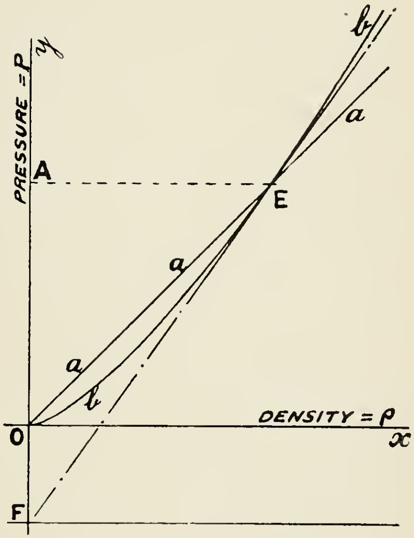


FIG. 161.

If we plot the analogous form of the adiabatic law, $P/\rho^\gamma =$ constant, Fig. 161 b, we no longer have a straight line diagram, but for small amplitude we may approximate by drawing a tangent EF cutting the axis of y at F. We may regard the point F as a new origin which will give the pressures proper to the limited portion of the curve approximated on the Boyle's law basis. From geometrical considerations we have AO to AF in the relation 1 is to γ, the relation of the real to the fictitious pressure of the gas; this at once gives us Laplace's correction.

In this case the assumption is obviously that the amplitude is small, for otherwise the tangent EF no longer approximates sufficiently to the actual curve.

The rationale of Laplace's correction may also be studied from the direct examination of the conditions. If we suppose in an adiabatic gas that a small isolated compression wave be constrained to move with the velocity proper to the gas obeying Boyle's law, the pressure during the reflection of the wave will be in excess of the momentum the wave communicates, to the extent that an adiabatic compression pressure is greater than the Boyle's law pressure for a given change of density. For small amplitude this is in the relation of γ to unity. Obviously the wave must travel faster to supply the momentum necessary to equalise, and since the momentum communicated *per unit time* varies as the *square of the velocity*, the velocity must be multiplied by $\sqrt{\gamma}$.

The question of the behaviour of an adiabatic wave of sensible amplitude is one of great complication that yet awaits a general solution. The compression regions are always endeavouring to move faster and the rarefaction regions slower than the mean velocity. From the present standpoint this is evidently due to the pressure increase becoming proportionately greater than the density increase (Fig. 161), and *vice versâ*, thus destroying the necessary balance between the pressure reaction and the communication of momentum by which it is maintained. The more usual and equally correct point of view is to attribute the difference of velocity of different portions of the wave to the difference of temperature of its parts.

Where we have a train of waves in a gas following the adiabatic law, it has been shown that there must be a pressure increase due to the energy that enters the thermodynamic system. Where the train is continuous, as in the Kundt's tube, no complication arises from this cause, but where we are dealing with a limited train, it is difficult to see in what manner this pressure can be confined to the region occupied by the wave train; according to thermodynamic principles it must be distributed uniformly and press equally in every direction. If this is true, the wave train as a whole will expand, and the remainder of the fluid will be compressed, so that the mean density of the wave train *will become less than that of the undisturbed fluid.* On this basis, employing the principle of § 5, a wave train under the conditions we are now supposing must be regarded as conveying *negative momentum.*

ADDENDUM B.

In an article on radiation in the "Encyclopædia Britannica,"[1] Larmor gives a theorem which purports to be a general proof of the transmission or communication of momentum by wave motion. Poynting[2] has given a condensed edition of this alleged proof, which may be quoted, as follows :—

"Let us suppose that a train of waves is incident normally "on a perfectly reflecting surface. Then, whether the reflecting "surface is at rest, or is moving to or from the source, the perfect "reflection requires that the disturbance at its surface shall be "annulled by the superposition of the direct and reflected trains. "The two trains must therefore have equal amplitudes. Suppose "now that the reflector is moving forward towards the source. "By Döppler's principle the waves of the reflected train are "shortened, and so contain more energy than those of the incident "train. The extra energy can only be accounted for by supposing "that there is a pressure against the reflector, that work has to "be done in pushing it forward. . . . A similar train of reasoning "gives us a pressure on the source, increasing when the source "is moving forward, decreasing when it is receding."

[1] Vol. xxxii., p. 121 (b).
[2] Pres. Address, Phys. Soc., *l.c. ante.*

Now it is evident that the whole of this reasoning rests on the assumption that the reflector, while impervious to the waves, is freely pervious to the medium;[1] an assumption that may be true in the case of light, but is certainly not true in the case of sound.

Poynting evidently appreciates this difficulty, for he says:—

"It is essential, I think, to Larmor's proof that we should be "able to move the reflecting surface forward without disturbing "the medium except by reflecting the waves." But further on he says:—

"But for sound waves I venture to suggest a reflector which "shall freeze the air just in front of it, and so remove it, the "frozen surface advancing with constant velocity u. Or perhaps "we may imagine an absorbing surface which shall remove the "air quietly by solution or chemical combination."

Now this is the first time that the author has heard it seriously suggested that portions of any dynamic system, essentially involved in that system, may be stolen away without affecting the sequence of events; it is, at least, evident that any such assumption totally invalidates Larmor's theorem as a generalisation, and in particular in its application to ordinary dynamic wave motion. It is very surprising to find that Poynting subsequently states that he finds Larmor's proof quite convincing.

In the address from which the above quotations have been given, Poynting cites an experiment by Prof. Wood intended to demonstrate the reality of sound pressure. In this experiment the sound waves from a strong induction-spark are focussed by a concave reflector on to a set of vanes as used on a radiometer, causing them to spin round. Now it is fair to assume that the cause of the emission of sound waves by an induction-spark is the heating of the air suddenly and locally by the spark energy, and consequently the wave will primarily be a compression wave. If steps were taken to cool the air immediately after it

[1] This fact is mentioned by Rayleigh, *Phil. Mag.*, vol. iii., 1902, p. 338.

had been heated, doubtless a rarefaction wave of equal displacement would follow, but no such steps are taken. It is true that the air initially heated by the spark is rapidly cooled by giving up its heat to the surrounding air, but this expands the air to which the heat is passed on, so that, on the principle of the author's bottle calorimeter,[1] no loss of volume takes place. There is possibly some minute quantity of heat lost to the conductors by which the current is supplied to the spark, but except for this the waves emitted will, on the whole, be compression waves involving a displacement of matter, and carrying the momentum appropriate to the mass displaced travelling with the velocity of sound. Ultimately the heated air is carried away by convection, but this does not affect the problem.

It is therefore evident that this experiment proves nothing, except that which we know already, *i.e.*, a displacement of matter carries with it momentum.

It is probable that other more or less successful experiments designed to demonstrate the existence of sound pressure involve some similar fallacy. It must be borne in mind that an unsymmetrical design of sound generator may conceivably emit pressure waves containing momentum in one or more directions, and rarefaction waves in others, or perhaps the air displaced by the pressure waves emitted in one direction may be replaced by a steady flow in other directions. On the other hand, it is possible that by some highly refined method the true pressure of a continuous wave train may be detected and measured, and the theoretical result that it is due to the energy passed into the thermodynamic system may some day receive confirmation.

ADDENDUM C.

In the foregoing Appendix and Addenda A and B the assumption has been made that the change of mean pressure within an enclosure containing a perfect gas is directly proportional to the

[1] Addendum C.

heat added or taken away, and mention has been made of a form of calorimeter proposed by the author depending upon this principle.

It is evident that if the principle can be proved as a general proposition as relating to the total heat it is also proved in relation to heat differences, that is heat added or subtracted.

The following proof goes beyond the problem as presented by the calorimeter, and applies generally for an enclosure in which the various portions of the gas are artificially constrained to occupy given positions *by any means whatever*, including, for example, the case of a wave train or other dynamic disturbance.

Let the enclosure be supposed divided into a number of small equal elements, and, examining firstly the conditions that apply to each small element to which it may be supposed that a quantity of heat h is supplied and distributed uniformly, giving rise to a uniform pressure P and temperature T, we have :—

$$\frac{P}{\rho} = T \times \text{const.}$$

but for a perfect gas

$$T = \frac{m}{h} \times \text{const.}$$

where m is the mass of the contents, hence

$$\frac{P}{\rho} = \frac{h}{m} \times \text{const.}$$

and since

$$\rho = \frac{m}{l^3}$$

$$P = \frac{h}{l^3}$$

for the element with which we are concerned.

Now, let n = the number of elements into which the enclosure is divided.

,, $P_1\ P_2\ P_3$ etc., be the pressures developed in the different elements to which quantities of heat $h_1\ h_2\ h_3$ have been supplied.

,, H = the total heat.

,, P_m = the resulting mean pressure.

Then the value of l^3 for each element will depend upon the number of elements into which the enclosure is divided, so that $l^3 \propto \frac{1}{n}$, and thus

$$P_1 = k\, n\, h_1$$
$$P_2 = k\, n\, h_2$$
$$P_3 = k\, n\, h_3 \text{ etc.}$$

where k is a constant.

$$\therefore \quad P_1 + P_2 + P_3 + \text{etc.} = k\, n\, (h_1 + h_2 + h_3 + \text{etc.})$$

But $$\frac{P_1 + P_2 + P_3 + \text{etc.}}{n} = P_m$$

and $h_1 + h_2 + h_3$ etc. is total heat added $= H$

$$\therefore \quad P_m = k\, H$$

this result continues to apply when the number of elements n becomes indefinitely great, hence the proposition is proved.

ADDENDUM D.

A Retrospective Note.

It is perhaps of some interest to state that the investigations included in the present appendix were actually made in the early part of 1905; the portion relating to the theory of sound momentum was submitted in the form of a draft paper to Professor Poynting, then President of the Physical Society, with whom the author had some correspondence on the subject.

The author did not receive sufficient encouragement to think it worth while submitting the paper, especially in view of previous experience and of the fact that not only Poynting, but Larmor, and at that time Rayleigh, were thoroughly identified with the general doctrine of sound momentum.

Referring to a warning note raised by the author, and with regard to the suggested paper, Professor Poynting wrote on June 9th, 1905, "Yes, I am quite sure about my views. But it is "quite evident that we are not going to see in the same direction. "I shall probably send my proof of pressure to the Physical

"Society some time so as to let those interested have their "choice."

Neither paper materialised. Lord Rayleigh shortly afterwards published his article (*loc. cit. ante*) in the *Phil. Magazine*, somewhat modifying his earlier conclusions,[1] and anticipating publication by the author in respect of two of the results now stated, *i.e.*, (1) the absence of momentum in or pressure due to a wave train under the conditions of Boyle's law; (2) the pressure of sound waves in a real gas as due to energy entering the thermodynamic system.

Before going to press the author submitted the above addendum to Prof. Poynting, and received the following reply, October 7th, 1907:—

"I stick to the postcard and have no objection to its publica- "tion."

"My proof of pressure was practically identical with Rayleigh's "and gave the result (1), and therefore I suppose (2). That is "why the paper did not materialise."

This is a truly astonishing statement in view of certain correspondence and MSS. in the author's possession. The following quotations are given as throwing some light on Prof. Poynting's actual position at the time in question.

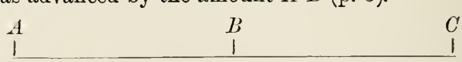
In a letter dated June 7th, 1905, referring to a draft MS.[2] submitted by the author, Poynting says:—

"On p. 3 the paragraph marked wants, I think, a few words "inserting to make it clear."

[1] Compare *Phil. Mag.*, vol. iii., 1902, pp. 341, 342 (Eq. 14).

[2] The portion of the author's draft paper referred to is as follows:—

"Then, let $A\,B$ be any length in the direction of propagation, at any instant, occupied by a train of waves. Let $B\,C$ be the place occupied by the same train when it has advanced by the amount $A\,B$ (p. 3).



Then by (4) $A\,B = B\,C$, and by (3) the particle at B, when waves occupy $A\,B$, is identical with particle at B when waves occupy $B\,C$; therefore the train contains the same mass of fluid as that of an equal volume of undisturbed fluid.

"Say thus:—Divide the wave train into lengths each con-"taining unit mass. The time taken by each of these lengths to "pass a given point is proportional to the length. Also the time "during which, etc. . . . or something of the sort. But it "appears to me that this is not the whole story, *but that the* "*motion does communicate momentum.* If the velocity forward is "$u \longrightarrow$, momentum $\rho u \times u = \rho u^2$ crosses the plane $\longrightarrow$. If the "velocity is $u \longleftarrow$, momentum ρu^2 crosses the plane $\longleftarrow$, and "both of these give an *addition* of momentum $\longrightarrow$ to the region "on the forward side of the plane." [1]

For the purpose of reference p. 3 of the author's original MS. is given in the accompanying footnote, the paragraph marked by Prof. Poynting being italicised. The initial and final paragraphs are completed as on pp. 2 and 4 of the MS.

Considering now a supposititious wave in a medium obeying Boyle's law,—

The volume occupied by any small unit of mass is by Boyle's law inversely as the pressure. Therefore the linear distance in the direction of propagation occupied by any small unit of mass is inversely as its pressure.

But the time during which pressure acts across the imaginary plane is by (4) *proportional to this linear distance. Therefore the time during which any pressure acts across the imaginary plane is inversely as that pressure, or $p\ t$ = constant for any small unit of mass. But $p\ t$ is the momentum communicated across the imaginary plane by pressure per unit area, and we have shown the total units of mass in any wave is the same as in undisturbed air. Consequently in a plane wave in a fluid obeying Boyle's law the momentum communicated by the pressure of the wave is precisely that communicated by the undisturbed fluid.*

And since the sum of the translation of mass ($m\ l$) by the wave is zero, the sum of the communication of momentum by motion ($m\ v$) is also zero.

That is to say, the plane wave in an elastic fluid obeying Boyle's law carries no momentum.

If the adiabatic wave is examined by the foregoing method an excess of mean pressure is found to exist, and without doubt, if the source of sound emits a continued succession of waves, momentum accompanies such waves as an ever-spreading field of excess mean pressure, but it is not clear that if the source ceases to emit, this pressure region will be confined to and move with the advancing waves; it appears more probable to the author that the air contained within the wave sphere shares in the excess pressure."

[1] This argument appears to involve a fallacy similar to that mentioned in footnote, p. 399. The note in question is the answer to an argument actually used by Poynting in conversation with the author.

It would appear from the above transcript that Poynting unquestionably held the view on June 7th, 1905, that a Boyle's law wave train would give rise to pressure increase or momentum transference, just as he held this view at the time of his address to the Physical Society in February of that year. Furthermore, it is evident that, at the time in question, result (2) was not a consequence anticipated by Poynting, for in another communication about the same date in reply to the author he says :—

"I have not thought of the sound pressure as accounted for "by the kinetic theory of gas. S. Tolver Preston, I think, did "so somewhere. It appears to me best in the first place to get "at the idea as I have done in the paper[1] as resulting from "known observable properties. Then go to the kinetic theory if "you like."

"The perfectly elastic solid—if by that is meant one that "obeys Hook's law rigidly—would give pressure apparently "from Larmor's theorem."[2]

It is difficult to understand how Prof. Poynting can have been led to make so extraordinary a statement as that contained in his present letter in view of the facts above given, and the author trusts that he will see his way to give publicity to some adequate explanation.

[1] Presidential Address, Phys. Soc., *l.c. ante.*

[2] At this time Poynting evidently has no misgivings as to the soundness of Larmor's theorem, and therefore must still have supposed that a Boyle's law wave-train carries momentum, apart from the evidence already given.

APPENDIX III.

A NOTE ON DISCONTINUOUS MOTION.

In § 101 allusion has been made to the instability of a surface of kinetic discontinuity in an inviscid fluid, and at the same time the impossibility of such a surface breaking up into finite vortex filaments is pointed out.

Helmholtz[1] has suggested that the instability takes the form of a development of convolutions of the surface of discontinuity or *surface of gyration.* He says :—

" An infinitely extended plane surface uniformly covered with parallel straight [infinitesimal] vortical filaments might indeed continue stable, but where the least flexure occurs at any time the surface curls itself round in ever narrowing spiral coils, which continually involve more and more distant parts of the surface in their vortex."

It is, unfortunately, not easy to form a clear picture of the continued transition that the above implies, or even of the resulting system of flow. There would appear to be no doubt, however, that Helmholtz's view is substantially correct.

[1] " Sensations of Tone," Appendix VII., B II.

APPENDIX IV.

LEAPING OR BOUNDING FLIGHT.

Many of the smaller birds habitually fly at a considerably greater velocity than would be computed from the pressure-velocity tables (Tables IX. and X.) on the lines of § 187.

The means by which this is accomplished is instructive. The bird flies briskly for a short distance and then *closes its wings*, continuing its flight as a simple projectile, so that the total flight consists of alternations of active flight and projectile flight. The flight path under these conditions consists of a series of leaps, as given in Fig. 162, in which the thick lines represent the periods of active flight and the fine lines the periods when the wings are closed.

It is evident both from the form of the flight path and from the behaviour of the bird that the whole of the sustentation takes place while the wings are spread, and that during this period the wings actually sustain both the weight of the bird and the centrifugal component due to its curvilinear flight path, and the sum of these is the effective load on the wing area in the sense of §§ 185—187.

The present note is based on visual observation. The largest bird witnessed by the author as employing the leaping mode of flight is the green woodpecker (*Picus viridis*); the weight of this bird averages about six to seven ounces (180 grams). Larger birds, as, for instance, the partridge, glide with *wings outstretched* when not in active flight.

The greatest length of "leap" in proportion to the corresponding active period, noted by the author, is about 3 : 1 (Fig. 162, c).

In this instance the species of bird was not identified. This proportion means that the reaction sustained by the wings when in action is approximately four times the weight of the bird, on which computation the flight velocity should be about twice that proper to the actual weight and wing area measurement.[1]

It is difficult to assess accurately the speed of flight of a bird under any circumstances, and most of all under the conditions now under discussion. Travelling at somewhat over thirty miles

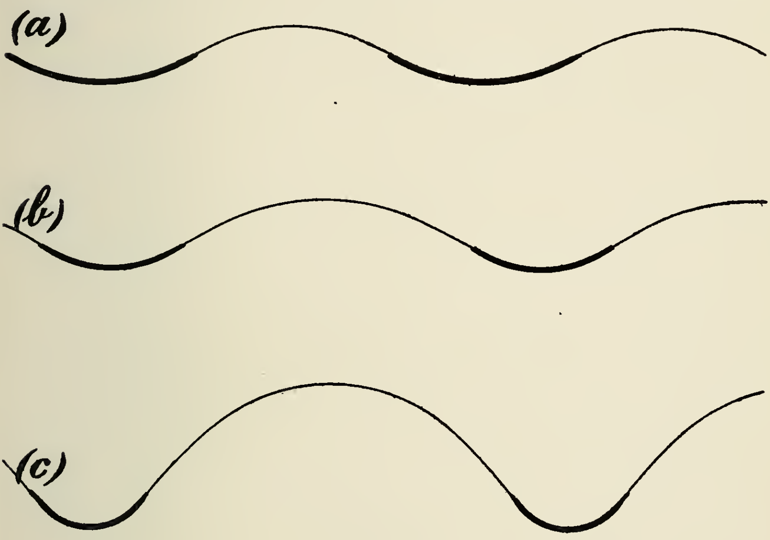


Fig. 162.

per hour on a motor vehicle, it is not an uncommon sight to see a *pied wagtail* or other small bird endeavouring to escape directly ahead by adopting the mode of flight under discussion. When hard pressed in this way the wagtail flies low, and its motion closely resembles the bouncing of an india-rubber ball on the

[1] It has already been pointed out (§ 187) that the problem is in all likelihood modified by the conditions of *active flight*, so that the tabulated figures, which relate to the gliding mode, may require to be multiplied by some unknown coefficient. In all probability the velocity of least resistance for a given bird in active flight is somewhere about 20 per cent. greater than for the gliding or soaring mode.

surface of the road, showing that the periods of active flight become very short in comparison with the length of the "leap."

Most of the smaller birds are able, by adopting the leaping mode of flight, to attain speeds of about thirty or forty miles per hour.

The probable reason for the leaping mode of flight being confined to the smaller birds is to be found in the considerations discussed in §§ 195, 196. The influence of aerofoil weight (wing weight) is less important in the case of a small aerodrome or bird than of a large one. Consequently nature can endow a small bird relatively with an extent of wing surface not "commercially" possible in the case of a larger bird, so that the smaller bird can, in normal active flight, fly slower than a large one, but by adopting the leaping mode it can, in effect, divest itself of its superfluous surface, and can then rival the larger birds in velocity. The leaping mode is, in fact, a means of adjustment, by means of which the conditions of least resistance can be approximated under considerable variations of velocity. If one of the larger birds, with its limited relative area, were to force its velocity up to the point at which leaping flight would pay, it would require an amount of energy per second far beyond its actual horse-power capacity.[1]

[1] *Ceteris paribus*, the horse-power of any animal or machine varies as the *square* of its linear dimension, whereas the weight varies as the *cube*. Thus the power per unit weight is greater for a small bird than a large one. (See "The Horse-power of the Petrol Engine in its Relation to Bore, Stroke, and Weight," "Proc. Inst. Automobile Engineers," April, 1907.)

Incidentally it may be remarked that it is probably for this reason that the soaring mode of flight, in which energy is captured from the wind, is principally employed by the larger birds, many of which are otherwise incapable of prolonged flight.

APPENDIX V.

SOARING.

AUTHORITIES are generally agreed at the present time that one at least of the varieties of soaring[1] practised by the larger birds involves the abstraction of energy from the wind fluctuation, that is to say, the soaring bird can derive the power required for its flight from the energy of turbulence of the wind (comp. §§ 37, 131).

It is clear that a bird having no horizontal force applied to it *from without* (in contradistinction to a kite which is connected to the earth by a string), is unable to effect any change in the total (horizontal) momentum of the air that comes within its grasp; consequently it cannot raise or lower the mean velocity of the wind, although it may be able to cause some parts to move faster and some more slowly.

It is evident that if a bird can, by altering its angle and altitude, so manipulate the wind coming within its grasp, that the portions that are moving in excess of the mean velocity have their velocity reduced, and those that are moving at less than the mean velocity are accelerated, the total energy of the

[1] Other methods of soaring are practised by many of the larger birds. In some cases soaring is accomplished by merely gliding on an up-current whose velocity is equal to or in excess of the rate of fall for gliding in still air; the up-current is sometimes due to the wind ascending the slope of a mountain or cliff, or may be due to the direct ascent of hot air from, for example, a sun-baked coast region.

Another form of soaring depends upon the proximity of masses of air having different velocities, as the live stream and "dead-water" region in the wake of an obstacle; the bird circles round and round, playing off the one mass of air against the other.

wind will be reduced, and the energy thus taken from the wind may become available for the purposes of propulsion. It is further evident that if a bird can carry the procedure suggested to the extent of reducing the whole of the air handled to a uniform velocity, that is to say, to its mean velocity, it will have taken away the whole of the energy that is available; *i.e.*, it will have removed the whole of the turbulence energy from the air within its reach. The foregoing assumes that the energy of turbulence consists wholly of motions in the direction of the main current, but the argument may, if required, be extended to include motions in the directions of the other two co-ordinate axes of space.

Without discussion of the means whereby the bird operates to play off one portion of the wind against another, we may, from the above considerations, form an outside estimate of the *available energy*. Thus if we prescribe some conventional form as representing the motion of turbulence, such as a simple harmonic motion in the line of flight, or a compound harmonic or circular motion of known velocity, we can calculate the turbulence energy per *unit volume*, and we may convert this into a thrust force per unit area of the stratum of air handled; if, then, we know the extent of this area in the case of any particular bird, and the weight of the bird, we can determine the gliding angle γ, the minimum value of which is a quantity otherwise known. Conversely we may, starting from the gliding angle and other data, determine the minimum velocity of turbulence on the convention chosen that will render soaring flight possible.

A question that presents some difficulty is the estimation of the *area of the stratum of air handled*. At first sight this might be supposed to be the "sweep" of the aerofoil, *i.e.*, $= \kappa A$ (§§ 109, 160), but the energy estimated on this basis from known fluctuation data appears to be insufficient.

The conception of the *peripteral area* (§ 210) suggests that, as in the case of the propeller blade, the cyclic or peripteral system may distribute the momentum over a much greater mass of fluid than that coming within the sweep of the aerofoil, and so a far larger mass of the air than that coming within the sweep area will be "handled" in the sense of the present discussion. On this basis the area of the stratum from which energy may be drawn is given by the expression, $\frac{1 + \epsilon}{1 - \epsilon} \kappa A$ (comp. § 210).

In the following example the turbulence velocity is computed necessary to provide the requisite energy to a hypothetical albatros, whose data are:—

Weight . . = 14 lbs.
Area . . . = 5 square feet.
n . . . = 12, hence $\kappa = 1{\cdot}195$ and $\epsilon = {\cdot}75$.
γ . taken = 1/7

The computation will be made both on the basis of *sweep* and that of *peripteral area*, and the figures will be given both for a simple harmonic motion and for circular motion, the assumption being in all cases that the whole of the available energy is utilised. As in all probability the bird can only utilise a comparatively moderate portion of the total available energy, the actual velocity of fluctuation will require to be very much greater than that stated in each case, in order that soaring should become possible.

Now resistance to flight = W γ, which from the foregoing data = 14 ÷ 7 = 2 pounds, or in absolute units = 64·4 poundals, or energy required per foot traversed = 64·4 ft. poundals.

Sweep = $\kappa A = 5 \times 1{\cdot}195 = 6$ (approx.), and mass of air handled (on basis of sweep) per foot traversed = ·078 = 6 = ·47.

If v be the velocity of mean square of turbulent motion, energy per foot traversed is

$$\frac{{\cdot}47 \times v^2}{2}$$

$$\therefore \quad {\cdot}47 \times v^2 = 64{\cdot}4 \times 2$$

$$\text{whence } v^2 = 274$$

$$\text{or} \quad v = 16{\cdot}5$$

Thus if the motion of turbulence is equivalent to a superposed circular motion, that is to say, if it consist of two component horizontal simple harmonic motions at right angles, and if the bird is able to abstract the total energy of both components, then v will be the maximum velocity of either component, or the uniform velocity of the equivalent circular motion; hence under the supposed conditions the maximum velocity of turbulence = 16·5 feet per second.

If the turbulence contain only one harmonic component, or if, which amounts to the same thing, the bird is only able to take advantage of the harmonic component in the line of flight, the available energy for a given maximum velocity will be only half that on the basis of circular motion; hence, in order that the necessary energy should exist in the wind, the maximum velocity must be multiplied by $\sqrt{2}$, or, on simple harmonic basis, the maximum velocity (plus or minus) of fluctuation becomes 23·4 feet per second.

The above estimates are on the basis of *sweep*. On the basis of *peripteral area* we have mass of air handled per foot traversed—

$$= \frac{1 + \epsilon}{1 - \epsilon} \rho \kappa A = \frac{1{\cdot}75}{{\cdot}25} \times {\cdot}47 = 3{\cdot}3 \text{ lbs.}$$

$$\therefore \quad \frac{3{\cdot}3 \times v^2}{2} = 64{\cdot}4$$

$$v^2 = 39, \text{ or, } v = 6{\cdot}25.$$

∴ on the basis of circular motion the maximum velocity of turbulence = 6·25 feet per second.

Or, on the simple harmonic basis, $6{\cdot}25 \times \sqrt{2} = 8{\cdot}8$ feet per second.

In the foregoing investigation the question of the *means* whereby the energy is trapped, or the possible percentage of the total that is available, is left untouched. The whole subject belongs essentially to the later portions of the work, *Aerodonetics*, where the matter will be treated more fully; the present publication is only made as an illustration of the employment of the peripteral theory expounded in the present work.

APPENDIX VI.

AN ELECTRO-MAGNETIC ANALOGY.

THE Eulerian theory of the inviscid fluid gives results that, it has already been remarked, bear but little resemblance to the behaviour of any actual liquid or gas. It is the more remarkable that these self-same results possess much that is in common with electrical phenomena. Thus the hydrodynamic plottings are true representations of the electrical and magnetic fields, and the theorem of energy and other Eulerian propositions in general apply.

The present analogy (for it is so far no more than an analogy) is one that has frequently attracted attention, and it is not without interest to follow the matter into the by-ways of hydrodynamic theory dealt with in the present work.

If we take the magnetic flux as the analogue of the flow (ψ function), then the electric current becomes a cyclic motion around the conductor. This point of analogy is emphasised by the need for a doubly or multiply connected region in both cases, in the case of the electric current for the completion of the circuit, and in the case of hydrodynamic theory in order that cyclic motion should become possible.[1]

If the conductor be situated in a magnetic field, it will experience a force at right angles to the direction of the field,

[1] The making or breaking of an electrical circuit alters simultaneously the connectivity of the regions both *internal* and *external* to the conductors; it is the latter that is the essential according to modern views, although it is the connectivity internal to the conductor that is usually present in the mind when reference is made to the completion of the circuit.

just as has been shown to exist in the case of the peripteral system, so that again we find the analogy holds. Thus, let us suppose a straight conductor in a uniform rectilinear magnetic field, the conductor and the lines of force being at right angles, and let the conductor be part of a completed circuit of zero resistance, carrying a current of some stated strength; then the conductor will experience a force at right angles to the direction of magnetic flux $= F$. Now let us apply a force F_1 equal and opposite to F, acting from without on the conductor, so that the latter will be held stationary; we may regard this force as the analogue of the weight of an aerodone supported in an Eulerian fluid, the electric current representing the cyclic component of the peripteroid motion, and the magnetic flux the superposed translation, in accordance with the *régime* of §§ 80 and 122.

If we suppose now a resistance to be inserted in the electrical circuit, the current, and therefore the force F, will tend to fall off, but the applied force F_1 continues, so that the conductor is set in motion in the magnetic field and is maintained in motion, the energy expended by the applied force F_1 being accounted for as energy lost in the electrical circuit; this is in fact the principle of the generation of an electric current by means of a dynamo.

The motion of the conductor under the influence of the force F_1 corresponds in our analogy to the descent of an aerodone in its gliding path, the gliding angle being represented by the velocity of the conductor divided by the velocity of the magnetic flux.

It is difficult to carry the present analogy much further without some stretching of the imagination or distortion of fact; even thus far there are many difficulties. For example, there is nothing in the hydrodynamic analogue of the electro-magnetic system depicted to give the conductor a *sense of direction* in the magnetic flux; its only knowledge of its motion through the supposed hydrodynamic stream is its *relative* motion, and as such it is difficult to see in what manner a conductor consisting of symmetrically disposed components (molecules) can distinguish between the real and apparent directions of the magnetic stream, in other words, how it can distinguish between an *impressed transverse motion* and a *transverse component* of the magnetic field. The analogy between the Eulerian fluid and the luminiferous ether is strong, but at present is not strong enough to bear any great weight.

In spite of difficulties, it appears probable to the author that in the near future some use may be made of existing electrical theory as an auxiliary means of investigating the aerodynamics of flight. Thus, in the general dynamics of the periptery, and in connection with the relations of the strength of the cyclic motion and the magnitude of the load reaction, it may be that mathematical solutions exist, ready to hand, in the analogous electrical theory, such as appropriately interpreted may some day be found to be of service.

APPENDIX VII.

FLUID RESISTANCE STUDIED BY THE AID OF AN IMPROVED KIND OF HYPOTHETICAL MEDIUM.

In § 131 a suggestion is made that leads to a new method of treatment of problems in fluid resistance.

Let us imagine a modification of the medium of Newton in which the particles, instead of being at rest, are in a state of agitation, and in the first instance let us suppose that all the particles, moving in directions at random, have the same velocity.

Taking first the case of a normal plane travelling at a velocity greater than that of the particles, we have the resistance proportional to the energy per unit volume (§ 131), the energy being reckoned only in respect of motion in the direction of flight, of either plus or minus sign. This energy is made up of two parts, the corpuscular energy of the medium, of which one third only counts as being in the direction of the axis of flight, and the energy of translation.

Now the corpuscular energy is constant in respect of the velocity of flight, and the energy of translation varies as the square of this quantity, consequently the law of resistance for this modified Newtonian medium will be, $P = k\,V^2 + n$, where k and n are constants.

If the velocity of the plane, instead of being greater than that of the particles, be less, the medium will exert a pressure on the back of the plane as well as on the face, and the resistance will be due to the pressure difference.

Taking the velocity as very low, then, the pressure being due to the bombardment of the particles, it may be easily demonstrated that the pressure difference and therefore the resistance must vary directly as the velocity. This may be regarded as the equivalent of "Stokes stage" in the case of a real fluid.

If the particles of the medium have different velocities the same general principles apply, only if the method is to be interpreted quantitatively the problem becomes a trifle more complex as involving the integration of a series of some kind.

In the case of a normal plane such as we have so far considered, the components of the motion of the particles transverse to the direction of flight have no influence. In the case of a solid body or curved lamina this is not the case, the lateral bombardment cannot be without effect on the total resistance.

Without examining the problem analytically, it appears obvious to the author that if (as is the case in a real gas) the energy of the particles is equally distributed in the three "degrees of freedom," that is in the directions of the three co-ordinate axes, the resistance at high velocities will not, in respect of the corpuscular energy, depend upon the form of the surface in presentation, but will depend upon the cross sectional area only; and any relief that can be obtained by rounding off or pointing the surface in presentation will take effect only in respect of the portion of the resistance that varies as V^2. That is to say, in the expression, $P = k V^2 + n$, giving easy entrance lines will diminish the constant k, but will have no influence on the value of the constant n.

The modified Newtonian medium of our present hypothesis resembles in many ways the perfect gas of kinetic theory, but differs in one very important respect. The molecules of a perfect gas are not only in a state of motion, but are undergoing frequent encounters one with another. Whether these encounters are due to gross impact or to some kind of action at a distance is immaterial from the point of view of the present discussion. The particles or corpuscles of the hypothetical medium have no magnitude, consequently they do not encounter one another, and therefore the medium has no continuity.

It is probable that the difference in the behaviour of air or any other gas, and the medium, will be least at very high and very low velocities; at intermediate velocities the present mode of treatment is unlikely to be of any utility. It seems possible that the present theory may find some application in relation to the flight of high velocity projectiles.

APPENDIX VIII.

PROPULSION BY SAILS.

It is scarcely necessary to point out that the peripteral theory set forth in the present work is capable of wider application than to the problems concerned in aerial flight.

The sailing boat, for example, offers a very promising field for the application of the peripteral principles of flight, and furnishes strong confirmation of the present theory. We may look upon the sailing boat, and especially the racing craft with its fin or deep keel, as an aerofoil combination in which the under-water and above-water reactions balance one another.

Laying on one side for subsequent consideration the part of the problem that relates to the heeling of the vessel and its stability, we may treat the matter in the first instance as if the under-water and above-water forces lie in one horizontal plane.

Under these conditions the problem resolves itself into an aerofoil combination in which the aerofoil acting in the air (the sail spread) and that acting under water (the keel, fin, or dagger plate) mutually supply each other's reaction.

The result of this supposition is evidently that the minimum angle at which the boat can shape its course relatively to the wind is the sum of the under and above-water gliding angles.

If the boat had no body (hull), and the conditions of our supposition be complied with, this reasoning shows that the minimum angle of the course relatively to the wind would be the sum of the γ for water and the γ for air, which is probably a degree or so less than 20 degrees, or rather less than two "points."

In practice, the two reactions (under and above water) not being in one plane, there is a resultant torque which has to be taken by the *moment of heel* due to the stability of the vessel. This results in a necessity for *added surface* and resistance due to the motion of the hull, both above and below water, especially the latter; the actual course is in consequence at a greater angle, about twice that stated even in the most carefully designed craft.

It seems to the author that by taking the present view many points hitherto but partially understood appear in a new light. For example, the bulging or filling of sails beyond the line of relative wind direction, a phenomenon well known to yachtsmen and other sailors, is the strict analogue of the arched section with dipping front edge of the aerofoil so amply demonstrated in the foregoing pages.

Further, the "dagger plate," the well-known expedient of the designer of light-draught racing craft, evidently "scores" over the ordinary centre-board by reason of its greater aspect ratio.

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